Fuglede conjecture and tilings in the field of p-adic numbers

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Outline

Introduction

I. Fuglede's conjecture in $\mathbb{R}^d$

- $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable of finite non-zero Lebesgue measure.
- Ω is said to be spectral if there exists a set $\Lambda \subset \mathbb{R}^d$ such that $\{\frac{1}{Leb(\Omega)^{1/2}} e^{2\pi i x\cdot\xi}\}_{\xi\in\Lambda}$ form a Hilbert basis of $L^2(\Omega)$.
- We say that the set Ω tiles $\mathbb{R}^d$ by translation, if there exists a set $T \subset \mathbb{R}^d$ such that $\{\Omega + t : t \in T\}$ forms a partition a.e. of $\mathbb{R}^d$, equivalently,

$$\sum_{t\in T} 1_\Omega(x-t) = 1, \qquad Leb - a.e..$$

Fuglede's conjecture

Ω is a spectral set if and only if it tiles $\mathbb{R}^d$.

- Fuglede 1974 : true if either Λ or T is a lattice.

II. Results in $\mathbb{R}^d$

The conjecture is not true for $d \geq 3$ (both direction). It is still **open for** $d = 1, 2$.

- Tao 2004 : "spectral $\Rightarrow$ tiling" is false for $d \geq 5$.
- Matolsci 2005 : "spectral $\Rightarrow$ tiling" is false for $d = 4$
- Matolsci and Kolountzakis 2006 : "spectral $\Rightarrow$ tiling" is false for $d = 3$
- Matolsci and Kolountzakis 2006 : "tiling $\Rightarrow$ spectral" is false for $d \geq 5$
- Farkas and Gy 2006 : "tiling $\Rightarrow$ spectral" is false for $d = 4$
- Farkas, Matolsci and Móra 2006 : "tiling $\Rightarrow$ spectral" is false for $d = 3$
- Iosevich, Katz and Tao 2003 : true for convex planer sets.
- Łaba 2001 : true for union of two intervals.
- Lagarias, Wang 1996, 1997 : all tilings of $\mathbb{R}$ must be periodic.

III. General setting

- G is a local compact abelian group with Haar measure $\mathfrak{m}$. For $x \in G$, $\Omega + x := \{y + x \in G : y \in \Omega\}$.
- A character of G is a group homomorphism $\chi : G \to S^1$, i.e. $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ and $\chi(0) = 1$. Denote by $\widehat{G}$ the dual group which consists of all the characters of G.
- A subset $\Omega \subset G$ of finite Haar Measure is said to be **spectral** if there exists a set $\Lambda \subset \widehat{G}$ which form a Hilbert basis of $L^2(\Omega)$. The set Λ is called a spectrum of Ω and (Ω, Λ) is called a spectral pair.
- We say that the set Ω **tiles** G by translation, if there exists a set $T \subset G$ such that $\{\Omega + t : t \in T\}$ forms a partition a.e. of G, equivalently,

$$\sum_{t \in T} 1_\Omega(x - t) = 1, \qquad a.e. \;\; x \in G.$$

  The set T is called a translate of Ω and (Ω, T) is called a tiling pair.

IV. Spectral set conjecture

Question : Ω is a spectral set if and only if it tiles G ?

- The case $G = \mathbb{Z}$ is open. The case $G = \mathbb{Z}/p^n\mathbb{Z}$ is true.
- The case $G = \mathbb{R}^d$ is the famous Fuglede conjecture.
- How about the case $G = \mathbb{Q}_p^d$? We confirm the conjecture when $d = 1$.

Fuglede's conjecture on $\mathbb{Q}_p$

I. The p-adic numbers and the topology

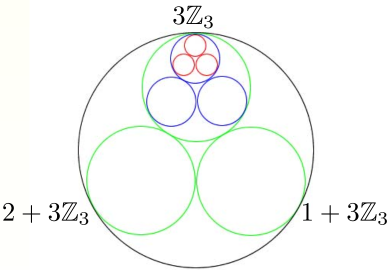
- Ring $\mathbb{Z}_p$ of p-adic integers :

$$\mathbb{Z}_p \ni x = \sum_{i=0}^{\infty} a_i p^i.$$

- Field $\mathbb{Q}_p$ of p-adic numbers : fraction field of $\mathbb{Z}_p$:

$$\mathbb{Q}_p \ni x = \sum_{i=v(x)}^{\infty} a_i p^i, \quad (\exists v(x) \in \mathbb{Z}).$$

Absolute value : $|x|_p = p^{-v(x)}$, metric : $d(x,y) = |x-y|_p$.



Remark : $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$. The topology on $\mathbb{Q}_p$ is the same as the topology of symbolic space.

II. Arithmetic in $\mathbb{Q}_p$

Addition and multiplication : similar to the decimal way.
"Carrying" from left to right.

Example : $x = (p-1) + (p-1) \times p + (p-1) \times p^2 + \cdots$, then

- $x + 1 = 0$. So,

$$-1 = (p-1) + (p-1) \times p + (p-1) \times p^2 + \cdots .$$

- $2x = (p-2) + (p-1) \times p + (p-1) \times p^2 + \cdots .$

We also have substraction and division.

III. Haar measure and characters on $\mathbb{Q}_p$

The Haar measure $\mathfrak{m}$ on $\mathbb{Q}_p$ is such that the unit ball $\mathbb{Z}_p$ has measure 1.

- A character $\chi \in \widehat{\mathbb{Q}_p}$ (where $\{x\} = \sum_{n=v_p(x)}^{-1} a_n p^n$) :

$$\chi(x) = e^{2\pi i \{x\}}.$$

  Notice : χ is a local constant function ($\chi(x) = 1$, if $x \in \mathbb{Z}_p$).
- For any $y \in \mathbb{Q}_p$, we define

$$\chi_y(x) = \chi(yx).$$

  The map $y \mapsto \chi_y$ from $\mathbb{Q}_p$ onto $\widehat{\mathbb{Q}_p}$ is an isomorphism.

IV. Fourier transformation on $\mathbb{Q}_p$

For $f \in L^1(\mathbb{Q}_p)$, the Fourier transform of f is defined to be

$$\widehat{f}(y) = \int_{\mathbb{Q}_p} f(x)\overline{\chi_y(x)}dx, \quad (dx = d\mathfrak{m}).$$

Examples :

- $\widehat{1_{B(0,p^\gamma)}}(\xi) = p^\gamma 1_{B(0,p^{-\gamma})}(\xi)$
- Let $\Omega = \bigsqcup_{j=1} B(c_j, p^\gamma)$. Then $\widehat{1_\Omega}(\xi) = p^\gamma 1_{B(0,p^{-\gamma})}(\xi) \sum_j \chi(-c_j\xi)$.

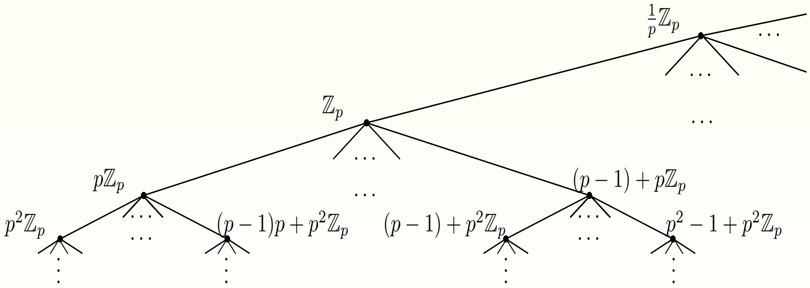
Lemma (A criterion of spectral set)

A Borel set Ω of finite Haar measure is a spectral set with Λ as a spectrum iff

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \sum_{\lambda\in\Lambda} |\widehat{1_\Omega}(\lambda - \xi)|^2 = \mathfrak{m}(\Omega)^2.$$

V. Tree structure of $\mathbb{Q}_p$

- Vertices $\mathcal{T}$: balls in $\mathbb{Q}_p$.
- Edges $\mathcal{E}$: pairs $(B', B) \in \mathcal{T} \times \mathcal{T}$ such that $B' \subset B$, $\mathfrak{m}(B) = p\mathfrak{m}(B')$, (denoted by $B' \prec B$).



VI. Bounded open sets in $\mathbb{Q}_p$

Any bounded open set O of $\mathbb{Q}_p$ can be described by a subtree $(\mathcal{T}_O, \mathcal{E}_O)$ of $(\mathcal{T}, \mathcal{E})$.

- Let B^* be the smallest ball containing O, which will be the root of the tree. For any given ball B contained in O, there is a unique sequence of balls $B_0, B_1, \cdots, B_r$ such that

$$B = B_0 \prec B_1 \prec B_2 \prec \cdots \prec B_r = B^*.$$

- The set of vertices $\mathcal{T}_O$ is composed of all such balls $B_0, B_1, \cdots, B_r$ for all possible balls B contained in O.
- The set of edges $\mathcal{E}_O$ is composed of all edges $B_i \prec B_{i+1}$ as above.

VII. Comapact open set in $\mathbb{Q}_p$

Any compact open set can be described by a finite tree, because a compact open set is a disjoint finite union of balls of same size. In this case, as in the above construction of subtree we only consider these balls of same size as B.

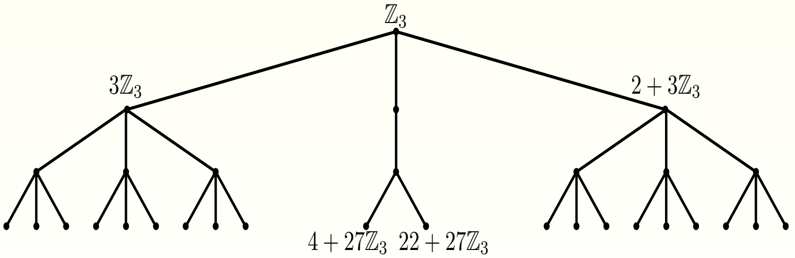


FIGURE : $\Omega = 3\mathbb{Z}_3 \sqcup (2+3\mathbb{Z}_3) \sqcup (4+27\mathbb{Z}_3) \sqcup (22+27\mathbb{Z}_3)$.

VIII. p-homogenous subsets in $\mathbb{Q}_p$

- A subtree $(\mathcal{T}', \mathcal{E}')$ is said to be homogeneous if the number of descendants of $B \in \mathcal{T}'$ depends only on $|B|$. If this number is either 1 or p, we call $(\mathcal{T}', \mathcal{E}')$ a p-homogeneous tree.
- An bounded open set is said to be homogeneous (resp. p-homogeneous) if the corresponding tree is homogeneous (resp. p-homogeneous).
- A bounded open p-homogenous set must be compact.

VIII. p-homogenous subsets in $\mathbb{Q}_p$

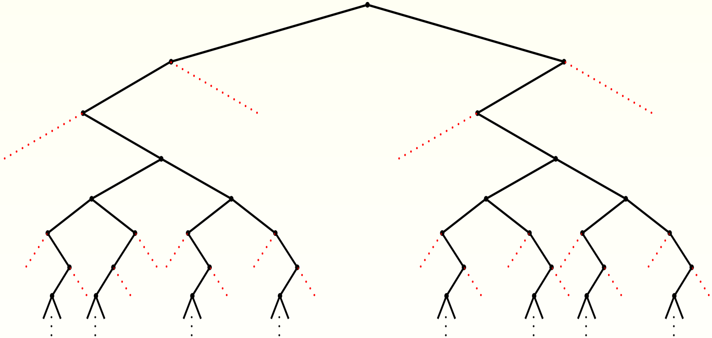


FIGURE : A 2-homogenous tree

VIII. p-homogenous subsets in $\mathbb{Q}_p$

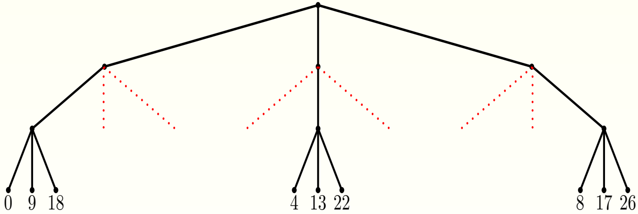


FIGURE : A 3-homogeneous tree

IX. Fuglede's conjecture on $\mathbb{Q}_p$

Theorem (A. Fan, S. Fan, L, R. Shi, arXiv :1512.08904)

A Borel set $\Omega \in \mathbb{Q}_p$ is a spectral set if and only if it tiles $\mathbb{Q}_p$. Moreover, Ω is an almost compact open set, and the corresponding compact open set is p-homogenous.

- A set $\Omega \subset \mathbb{Q}_p$ is called an almost compact open set, if $\exists$ compact open $\Omega' \subset \mathbb{Q}_p$ such that $\mathfrak{m}(\Omega \setminus \Omega') = \mathfrak{m}(\Omega' \setminus \Omega) = 0$.

Fuglede's conjecture for compact open set in $\mathbb{Q}_p$

I. Tree structure of $\mathbb{Z}/p^\gamma\mathbb{Z}$

We identify $\mathbb{Z}/p^\gamma\mathbb{Z} = \{0, 1, \cdots, p^\gamma - 1\}$ with $\{0, 1, 2, \cdots p - 1\}^\gamma$ which is considered as a finite tree, denoted by $\mathcal{T}^{(\gamma)}$.

- Vertices $\mathcal{T}^{(\gamma)}$: consists of the disjoint union of the sets $\mathbb{Z}/p^n\mathbb{Z}, 0 \leq n \leq \gamma$. Each vertex, except the root of the tree, is identified with a sequence $t_0 t_1 \cdots t_n$ with $0 \leq n \leq \gamma$ and $t_i \in \{0, 1, \cdots, p - 1\}$.
- Edges : consists of pairs $(x, y) \in \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^{n+1}\mathbb{Z}$ with $x \equiv y \mod p^n$, where $0 \leq n \leq \gamma - 1$.

For example, each point t of $\mathbb{Z}/p^\gamma\mathbb{Z}$ is identified with $t_0 t_1 \cdots t_{\gamma-1}$, which is a boundary point of the tree.

I. Tree structure of $\mathbb{Z}/p^\gamma\mathbb{Z}$

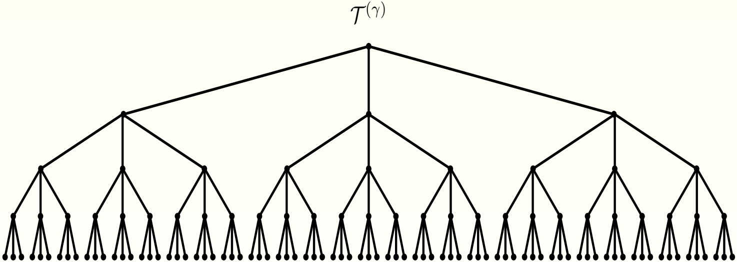


FIGURE : The set $\mathbb{Z}/3^4\mathbb{Z} = \{0, 1, 2, \cdots, 80\}$ is considered as a tree $\mathcal{T}^{(4)}$.

II. p-homogenous subsets $\mathbb{Z}/p^\gamma\mathbb{Z}$

Each subset $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ will determine a subtree of $\mathcal{T}^{(\gamma)}$, denoted by $\mathcal{T}_C$, which consists of the paths from the root to the points in C.
For each $0 \le n \le \gamma$, denote
$C_{\text{mod } p^n} := \{x \in \mathbb{Z}/p^n\mathbb{Z} : \exists\ y \in C, \text{ such that } x = y \bmod p^n\}$.

- Vertices $\mathcal{T}_C$: consists of the disjoint union of the sets $C_{\text{mod } p^n}, 0 \le n \le \gamma$.
- Edges : consists of pairs $(x, y) \in C_{\text{mod } p^n} \times C_{\text{mod } p^{n+1}}$ with $x \equiv y \mod p^n$, where $0 \le n \le \gamma - 1$.

The set C is called a p-homogenous subsets of $\mathbb{Z}/p^\gamma\mathbb{Z}$ iff the corresponding tree $\mathcal{T}_C$ is p-homogenous.

II. p-homogenous subsets $\mathbb{Z}/p^\gamma\mathbb{Z}$

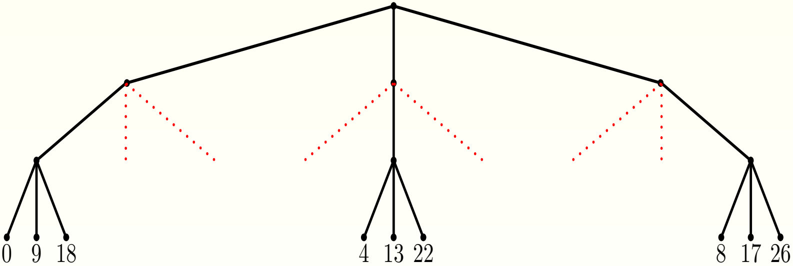


FIGURE : For $p = 3, \gamma = 3$, the tree p-homogeneous tree determined by $\{0, 4, 8, 9, 13, 17, 18, 22, 26\}$.

III. Spectral sets and tiles in $\mathbb{Z}/p^\gamma\mathbb{Z}$

Recall that the Fourier transform of a function f defined on $\mathbb{Z}/p^\gamma\mathbb{Z}$ is defined as

$$\widehat{f}(k) = \sum_{x \in \mathbb{Z}/p^\gamma\mathbb{Z}} f(x) e^{-\frac{2\pi i k x}{p^\gamma}}, (\forall k \in \mathbb{Z}/p^\gamma\mathbb{Z}).$$

Theorem (Fan–Fan–Shi, arXiv 2015) Let $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$. The following statements are equivalent.

(1) C is p-homogenous.

(2) There exists a subset $I \subset \{0, \cdots, \gamma\}$ such that $\sharp I = \log_p(\sharp C)$ and $\widehat{1_C}(p^\ell) = 0$ for $\ell \in I$.

(3) C is a spectral set in $\mathbb{Z}/p^\gamma\mathbb{Z}$, with

$$\Lambda = \left\{ \sum_{i \in I} a_i p^{-i-1} : a_i \in I \right\}.$$

(4) C tiles $\mathbb{Z}/p^\gamma\mathbb{Z}$, by

$$T = \left\{ \sum_{j \in J} a_j p^j : a_j \in \{0, \cdots p-1\} \right\}, \quad \text{where } J := \{0, \cdots, \gamma\} \setminus I.$$

III. Spectral sets and tiles in $\mathbb{Z}/p^\gamma\mathbb{Z}$

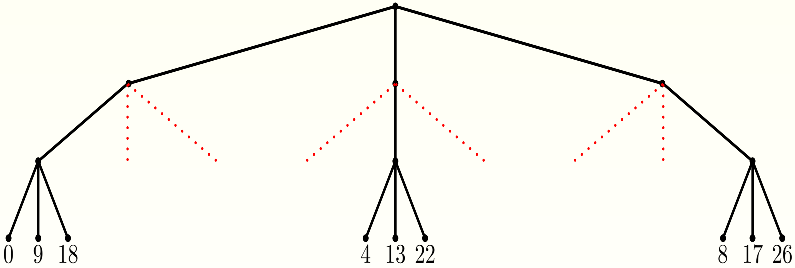


FIGURE : Here, $p = 3, \gamma = 3$, $I = \{1, 3\}, J = \{0, 2\}$.

IV. Compact open spectral sets $\mathbb{Q}_p$

W. l. o. g, we assume that Ω is of the form

$$\Omega = \bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p),$$

where $\gamma \geq 1$ is an integer and $C \subset \{0, 1, \cdots, p^\gamma - 1\}$.
Theorem (Fan–Fan–Shi, arXiv 2015) The following are equivalent.

(1) $\mathcal{T}_C$ is a p-homogenous tree.

(2) Ω is p-homogenous.

(3) Ω tiles $\mathbb{Q}_p$.

(4) Ω is a spectral set in $\mathbb{Q}_p$.

Proof of Fuglede's conjecture on $\mathbb{Q}_p$

I. Fourier transformation

A complex function f defined on $\mathbb{Q}_p$ is called *uniformly locally constant* if there exists $n \in \mathbb{Z}$ such that

$$f(x+u) = f(x) \quad \forall x \in \mathbb{Q}_p, \forall u \in B(0, p^n).$$

Lemma

Let $f \in L^1(\mathbb{Q}_p)$ be a complex-value integrable function.
(1) If f has compact support, then $\widehat{f}$ is uniformly locally constant.
(2) If f is uniformly locally constant, then $\widehat{f}$ has compact support.

A subset E of $\mathbb{Q}_p$ is said to be *uniformly discrete* if E is countable and $\inf_{x,y \in E} |x-y|_p > 0$.

Corollary

Let $\Omega \subset \mathbb{Q}_p$ be a Borel set of positive and finite Haar measure.
(1) If (Ω, Λ) is a spectral pair, then Λ is uniformly discrete.
(2) If (Ω, T) is a tiling pair, then T is uniformly discrete.

II. Convolution equation

Note that (Ω, T) is a tiling pair is equivalent the convolution equation

$$\sum_{t \in T} 1_\Omega(x - t) = 1, \qquad a.e. \;\; x \in G. \qquad \text{(Tilling)}$$

And (Ω, λ) is a spectral pair is equivalent to the following convolution equation

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \quad \sum_{\lambda \in \Lambda} |\widehat{1_\Omega}(\lambda - \xi)|^2 = \mathfrak{m}(\Omega)^2. \qquad \text{(Spectral)}$$

In general, we consider the convolution equation

$$\mu_E * f = 1,$$

where $\mu_E = \sum_{t \in E} \delta_t$ is a discrete measure, $0 \le f \in L^1(\mathbb{Q}_p)$, $\int_{\mathbb{Q}_p} f d\mathfrak{m} > 0$.

III. Distribution

The space $\mathcal{D}$ of Bruhat-Schwartz test functions is, by definition, constituted of uniformly locally constant functions of compact support.
A Bruhat-Schwartz distribution f on $\mathbb{Q}_p$ is by definition a continuous linear functional on $\mathcal{D}$.
The discrete measure μ_T is also a distribution : for any $\phi \in \mathcal{D}$,

$$\langle \mu_E, \phi \rangle = \sum_{\lambda \in E} \phi(\lambda).$$

The Fourier transform of a distribution $f \in \mathcal{D}'$ is a new distribution $\widehat{f} \in \mathcal{D}'$ defined by the duality

$$\langle \widehat{f}, \phi \rangle = \langle f, \widehat{\phi} \rangle, \quad \forall \phi \in \mathcal{D}.$$

IV. Zeros of a distribution

A point $x \in \mathbb{Q}_p$ is called a *zero* of a distribution f if there exists an integer n_0 such that

$$\langle f, 1_{B(y,p^n)} \rangle = 0, \quad \text{for all } y \in B(x, p^{n_0}) \text{ and all integers } n \leq n_0.$$

Denote by $\mathcal{Z}_f$ the set of all zeros of f.

Lemma (2)

Let E be a uniformly discrete set in $\mathbb{Q}_p$.
(1) If $\xi \in \mathcal{Z}_{\widehat{\mu_E}}$, then $S(0, |\xi|_p) \subset \mathcal{Z}_{\widehat{\mu_E}}$.
(2) The set $\mathcal{Z}_{\widehat{\mu_E}}$ is bounded.

Denote $n_f := \min\{n \in \mathbb{Z} : \widehat{f}(x) \neq 0, \text{ if } x \in B(0, p^{-n})\}$.

Corollary

If $\mu_E * f = 1$, then $\widehat{f}$ has compact support. The set $\mathcal{Z}_{\widehat{\mu_E}}$ is bounded and

$$B(0, p^{-n_f}) \setminus \{0\} \subset \mathcal{Z}_{\widehat{\mu_E}}.$$

Proof : Note that $\mu_E * f = 1$ implies $\widehat{\mu_E} \cdot \widehat{f} = \delta_0$.

V. Tiles are almost compact open

Suppose (Ω, T) is a tiling pair. Then

$$\mu_T * 1_\Omega = 1.$$

Then module a set of zero Haar measure,

$\Rightarrow T$ is uniformly discrete.

$\Rightarrow \widehat{1_\Omega}$ has compact support.

$\Rightarrow 1_\Omega$ is uniformly locally constant.

$\Rightarrow \Omega$ is a union of balls with the same radius.

Since Ω has finite Haar measure, we conclude that module a zero measure set, Ω is **compact-open**. Then "Tiling $\Rightarrow$ spectral " follows directly from the result of **Fan–Fan–Shi**.

VI. Spectral sets are tiles -I

Suppose (Ω, Λ) is a spectral pair. Then

$$\mu_\Lambda * \frac{|\widehat{1_\Omega}|^2}{\mathfrak{m}(\Omega)^2} = 1.$$

Then module a set of zero Haar measure,

$\Rightarrow \Lambda$ is uniformly discrete.

$\Rightarrow \widehat{|\widehat{1_\Omega}|^2}$ has compact support.

$\Rightarrow \Omega$ is bounded.

Without loss of generality, we assume that $\Omega \subset \mathbb{Z}_p$.

VI. Spectral sets are tiles -II

Recall that every sphere $S(0, p^{-n})$ either is contained in $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ or does not intersect $\mathcal{Z}_{\widehat{\mu_\Lambda}}$. Moreover, $B(0, p^{-n_f}) \setminus \{0\} \subset \mathcal{Z}_{\widehat{\mu_E}}$.

Let

$$\begin{aligned} \mathbb{I} : &= \left\{0 \le n < n_f : S(0, p^{-n}) \subset \mathcal{Z}_{\widehat{\mu_\Lambda}}\right\}, \\ \mathbb{J} : &= \{0 \le n < n_f : S(0, p^{-n}) \cap \mathcal{Z}_{\widehat{\mu_\Lambda}} = \emptyset\}. \end{aligned}$$

Take

$$U := \left\{ \sum_{j \in \mathbb{J}} \alpha_j p^j, \alpha_j \in \{0, 1, \ldots, p-1\} \right\}.$$

Then Ω is a tile of $\mathbb{Z}_p$ with tiling complement U. Then we can also tile $\mathbb{Q}_p$.

Fuglede's conjecture in $\mathbb{Q}_p^2$

Problem : Does Fuglede's conjecture hold in $\mathbb{Q}_p^2$?

Remark : Tiles and spectral sets are not necessarily almost compact open.
We partition $\mathbb{Z}_p$ into p Borel sets of same Haar measure, Set $S = \bigcup_{n=1}^{\infty} B(p^n, p^{-n-1})$, a union of countable disjoint balls, thus not compact open. Let

$$\begin{aligned} A_0 &= S \cup (B(1, p^{-1}) \setminus (1 + S)), \\ A_1 &= (B(0, p^{-1}) \cup B(1, p^{-1})) \setminus A_0, \\ A_i &= B(i, p^{-1}) \ \text{ for } \ 2 \leq i \leq p-1. \end{aligned}$$

Define

$$\Omega := \bigcup_{i=0}^{p-1} A_i \times B(i, p^{-1}) \subset \mathbb{Z}_p \times \mathbb{Z}_p.$$