Review of some results from Diophantine approximation

Alan Haynes

Dirichlet's theorem

Theorem: If α is a real number and N is a positive integer then we can find a rational number a/n with $1 \leq n \leq N$ and

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{1}{n(N+1)}.$$

Corollary: For every $\alpha \in \mathbb{R}$,

$$\inf_{n \in \mathbb{N}} n\|n\alpha\| = 0,$$

where $\| \cdot \|$ denotes the distance to the nearest integer.

Hurwitz's theorem

Theorem: For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ there are infinitely many $a/n \in \mathbb{Q}$ with

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{1}{\sqrt{5}n^2}.$$

Equivalently: For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there are infinitely many $n \in \mathbb{N}$ for which

$$n\|n\alpha\| \leq \frac{1}{\sqrt{5}}.$$

Badly approximable numbers

A real number α is **badly approximable** if there exists a constant $c(\alpha) > 0$ such that

$$\inf_{n \to \infty} n \| n\alpha \| \geq c(\alpha).$$

We write $\mathcal{B}$ for the set of all badly approximable numbers. A real number which is not badly approximable is called **well approximable.**

Borel, Bernstein (1909, 1912): $|\mathcal{B}| = 0$.

Jarnik (1926): $\dim \mathcal{B} = 1$.

Khintchine's theorem

Theorem: If $\psi : \mathbb{N} \to [0, \infty)$ is monotonic and if

$$\sum_{n \in \mathbb{N}} \psi(n) = \infty,$$

then for a.e. $\alpha \in \mathbb{R}$ there are infinitely many $a/n \in \mathbb{Q}$ for which

$$\left| \alpha - \frac{a}{n} \right| \leq \frac{\psi(n)}{n}.$$

If the sum above converges then for a.e. α the above inequality has only finitely many solutions $a/n \in \mathbb{Q}$.

Dirichlet's theorem in higher dimensions

Let $L : \mathbb{R}^d \to \mathbb{R}^{k-d}$ be a linear map, defined by a matrix with entries $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$, so that

$$L(x) = (L_1(x), \ldots, L_{k-d}(x)),$$

with

$$L_i(x) = \sum_{j=1}^{d} \alpha_{ij} x_j,$$

for $1 \leq i \leq k-d$. Then for any $N \in \mathbb{N}$, there exists an $n \in \mathbb{Z}^d$ with $|n| \leq N$ and

$$\|L(n)\| \leq \frac{1}{N^{d/(k-d)}},$$

where for $y \in \mathbb{R}^s$,

$$|y| = \max_{1 \leq i \leq s} |y_i| \quad \text{and} \quad \|y\| = \max_{1 \leq i \leq s} \|y_i\|.$$

Cassels's transference principle: homogeneous to inhomogeneous approximation

Theorem: Given a linear map L as above, the following statements are equivalent:

(T1) There exists a constant $C_1 > 0$ such that

$$\|L(n)\| \geq \frac{C_1}{|n|^{d/(k-d)}},$$

for all $n \in \mathbb{Z}^d \setminus \{0\}$.

(T2) There exists a constant $C_2 > 0$ such that, for all $\gamma \in \mathbb{R}^{k-d}$, the inequalities

$$\|L(n) - \gamma\| \leq \frac{C_2}{N^{d/(k-d)}}, \quad |n| \leq N,$$

are soluble, for all $N \geq 1$, with $n \in \mathbb{Z}^d$.

Badly approximable systems of linear forms

With a view towards eventually applying the transference principle, let $\mathcal{B}_{d,k-d}$ denote the collection of numbers $\alpha \in \mathbb{R}^{d(k-d)}$ with the property that there exists a constant $C = C(\alpha) > 0$ such that, for all nonzero integer vectors $n \in \mathbb{Z}^d$,

$$\|L(n)\| \geq \frac{C}{|n|^{d/(k-d)}}.$$

We refer to the elements of the set $\mathcal{B}_{d,k-d}$, as well as the systems of linear forms which they define, as collections of **badly approximable systems of linear forms.**

Khintchine-Groshev Theorem

Theorem: If $\psi : \mathbb{N} \to [0, \infty)$ is monotonic and if

$$\sum_{m \in \mathbb{N}} m^{d-1} \psi(m)^{k-d} = \infty,$$

then for a.e. $\alpha \in \mathbb{R}^{d(k-d)}$ there are infinitely many $n \in \mathbb{Z}^d$ with the property that

$$\|L(n)\| \leq \psi(|n|).$$

On the other hand, if this sum converges then for a.e. α there are only finitely many solutions to this inequality.

Size of the sets $\mathcal{B}_{d(k-d)}$

Khintchine-Groshev $\Rightarrow$ $|\mathcal{B}_{d(k-d)}| = 0$

Theorem (Schmidt, 1969): $\dim \mathcal{B}_{d(k-d)} = d(k-d)$

A note related to cut and projet sets

In our applications to cut and project sets we will sometimes be working with linear forms $L : \mathbb{R}^d \to \mathbb{R}$ which have the degenerate property that $L(\mathbb{Z}^d) + \mathbb{Z}$ is a periodic subset of $\mathbb{R}/\mathbb{Z}$. If we define $\mathcal{L} : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ by

$$\mathcal{L}(n) = L(n) \bmod 1,$$

then we can phrase this property by saying that the kernel of the map $\mathcal{L}$ is a nontrivial subgroup of $\mathbb{Z}^d$. For the types of cut and project sets that we want to understand, there is no way to avoid this degeneracy, but we will still want to be able to say something meaningful about the Diophantine approximation properties of L.

Relatively badly approximable linear forms

Let $S \leqslant \mathbb{Z}^d$ be the kernel of the map $\mathcal{L}$ from the previous slide, and write $r = \mathrm{rk}(S)$ and $m = d - r$. We say that L is **relatively badly approximable** if $m > 0$ and if there exists a constant $C > 0$ and a group $\Lambda \leqslant \mathbb{Z}^d$ of rank m, with $\Lambda \cap S = \{0\}$ and

$$\|\mathcal{L}(\lambda)\| \geq \frac{C}{|\lambda|^m} \quad \text{for all} \quad \lambda \in \Lambda \setminus \{0\}.$$

Remark: If L is relatively badly approximable, then the group Λ in the definition may be replaced by any group $\Lambda' \leqslant \mathbb{Z}^d$ which is complementary to S.