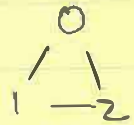
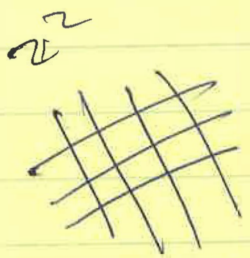
jw Brian Marcus.

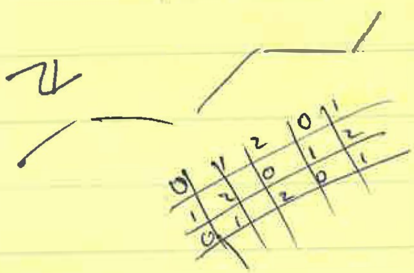


H – undirected graph.



G – undirected graph without self-loops.

$$\mathrm{Hom}(G,H) := \{x : G \to H \mid i \sim_G j \Rightarrow x_i \sim_H x_j\}$$

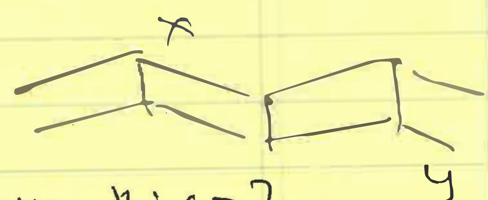


H – 3 (0, 1 – 2) – proper 3-colouring of ~~$\mathbb{Z}^2$~~ G.

H – 0 — 1 – ~~Hard~~ no two 1s are adjacent (hard core model)

$$H_{walk} = (\mathrm{Hom}(\mathbb{Z},H), \underbrace{E_{walk}}_{})$$

$$E_{walk} = \{(x,y) : x_i \sim y_i \ \forall i \in \mathbb{Z}\}.$$



$\boxed{\text{Main Qn: When is } \mathrm{diam}(H_{walk}) < \infty \text{ ?}}$

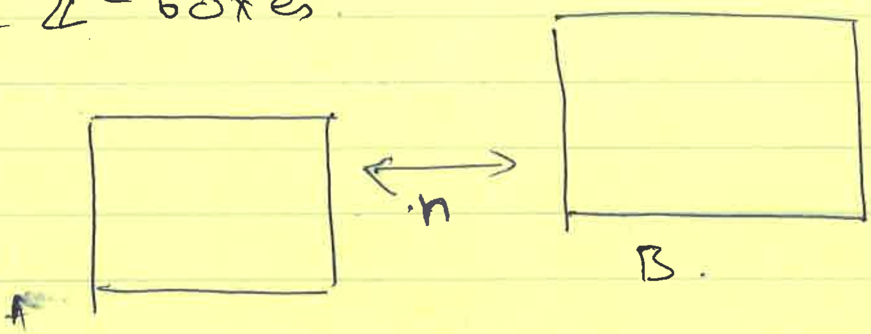
Motivation: $\mathrm{Hom}(\mathbb{Z}^2,H)$ forms a dynamical system (translation of say $x \in \mathrm{Hom}(\mathbb{Z}^2,H)$ is still a homomorphism).

Want to address → How do properties of H reflect in the dynamics of $\mathrm{Hom}(\mathbb{Z}^2,H)$?

By properties $\rightsquigarrow$ mixing properties.

What is a mixing property?

$A, B \subset \mathbb{Z}^2$ boxes.



$a \in \mathrm{Hom}(A, \mathcal{H})$, $b \in \mathrm{Hom}(B, \mathcal{H})$

(Does there exists $x \in \mathrm{Hom}(\mathbb{Z}^2, \mathcal{H})$ s.t. $x|_A = a$, $x|_B = b$?) $(*)$

n depends on ~~a~~ a, ~~B~~ b $\longrightarrow$ ~~tra~~ transitivity

$(*)$ $\rightarrow$ holds for all $\mathcal{H}$-connected

n independent of ~~A, B, A, B~~ a, b $\rightarrow$ block-gluing.

Qn: When is $\mathrm{Hom}(\mathbb{Z}^2, \mathcal{H})$ block-gluing?

Note if $\mathcal{H}$ is bipartite.

a, b are ~~sin~~ patterns on A, B singletons

then n depends on whether a, b

are of the same partite class.

So $\mathcal{H}$-bipartite $\Rightarrow$ $\mathrm{Hom}(\mathbb{Z}^2, \mathcal{H})$ not block-gluing

n independent of A, B. — phased block-gluing
(might depend on phase of a, b.)

$Hom(\mathbb{Z}^2, H)$ is ~~block~~ phased block-gluing iff

$$\operatorname{diam}(H_{walk}) < \infty$$

$\boxed{\text{Recall}}$ ($Hom\ \mathbb{Z}, H$.

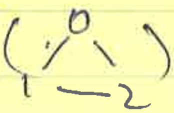
<u>Examples</u>:

$(012)^\infty$

0
1
2
0
1
2
0
1
2
⋮

Anything at finite distance ~~essentially look like this~~ to $(012)^\infty$ looks like $(012)^\infty$

— $\operatorname{distance}((012)^\infty, (01)^\infty) = \infty$.

$\operatorname{diam}(($  $)_{walk}) = \infty$

H

x_1 0

x_2 0

$\vdots$

x_n 0

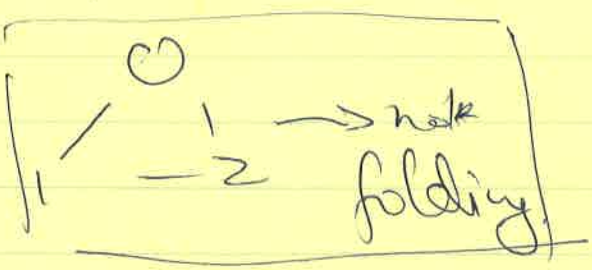
$0^\infty \sim x$ for all $x \in \mathrm{Hom}(\mathbb{Z}, H)$

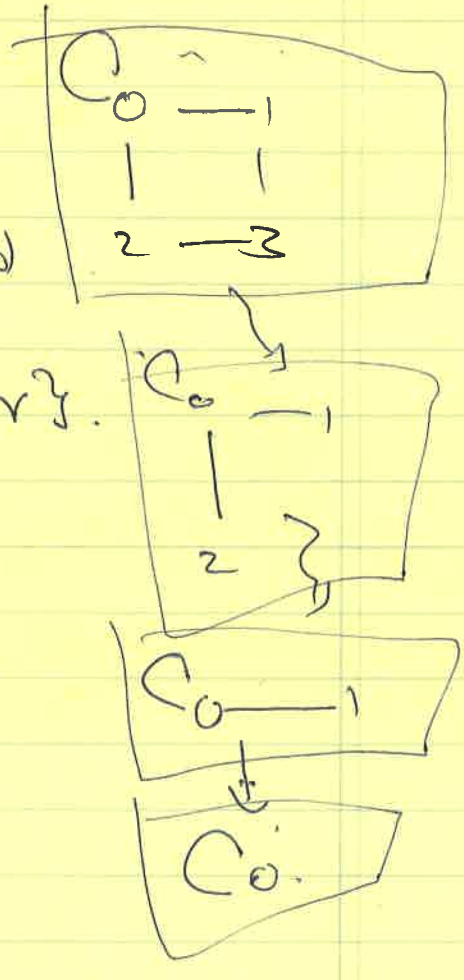
$\mathrm{diam}(0 - 1)_{walk} = 2$

More generally, H-graph.

v folds into w if $N_H(v) \subset N_H(w)$

$\{u \in H : u \sim_H v\}$.





$x \in \mathrm{Hom}(\mathbb{Z}, H)$

shift of $x \sim x$

R

x_2	x_1
x_1	x_0
x_0	x_{-1}
x_{-1}	x_{-2}
x_{-2}	x_{-3}.

Can replace all v's by w to get $y \sim x$ where $y \in \mathrm{Hom}(\mathbb{Z}, H \setminus \{v\})$

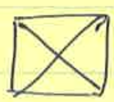
H is called bipartite dismantlable if $\exists$ sequence of folds starting with H and ending with $\bullet\!-\!\bullet$ or

Thm: H bip. dismantlable $\Rightarrow$ $\mathrm{diam}(H_{walk}) < \infty$

Thm: ~~H has~~ Let H be undirected graph without self loops and copies of C_4 ~~...~~ $\square$

then $\mathrm{diam}(H_{walk}) < \infty$ iff H is not ~~bip~~ bip. dismantlable.

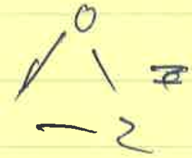
Remark: Converse is not true in general.

$\rightarrow$ K_4  $= H$ $\mathrm{diam}(H_{walk}) \leq 6$.

$\rightarrow$ Ronnie Pavlov

Conjecture: It is undecidable whether $\mathrm{diam}(H_{walk}) < \infty$.

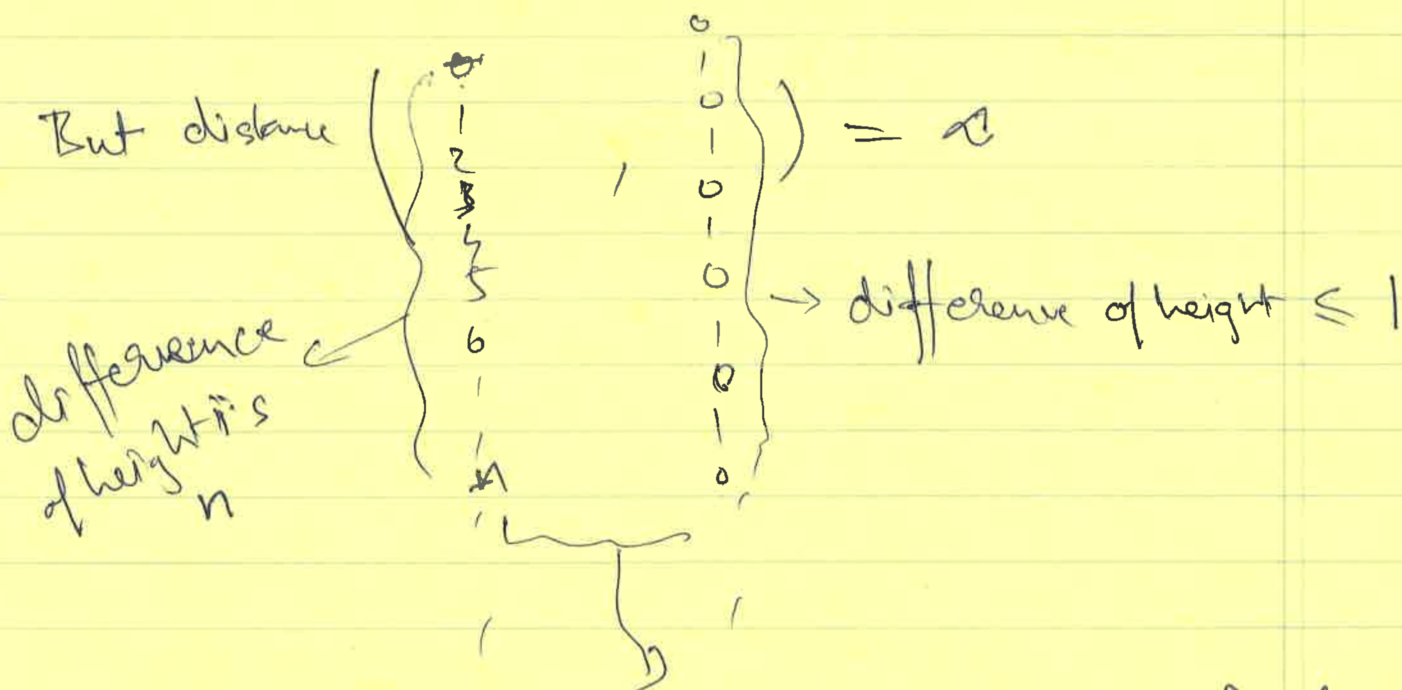
How to prove. $\operatorname{diam}(H_{walk}) = \infty$

for $H =$ 

There is a natural map. $\bmod 3 : \mathbb{Z} \to H$

This induces a covering map from

$(\mathbb{Z})_{walk}$ to $(H)_{walk}$.



number of steps $\geq \frac{n}{2}$ for all n.

$\Rightarrow$ $\operatorname{diam}((\mathbb{Z})_{walk}) = \infty$

$\Rightarrow$ $\operatorname{diam}(H_{walk}) = \infty$.

If H ~~is~~ has no self loops and four cycles use π: universal cover of $H \to H$.

Thm: It is decidable whether $\mathrm{diam}(H_{\mathrm{walk}}) \leq n$

Conjecture: It is undecidable whether

$$\mathrm{diam}(H_{\mathrm{walk}}) < \infty.$$