# An aperiodic set of 11 Wang tiles

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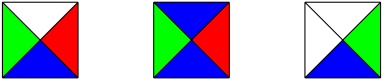
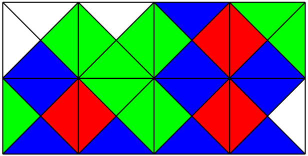
A Wang tile is a square tile with a color on each border

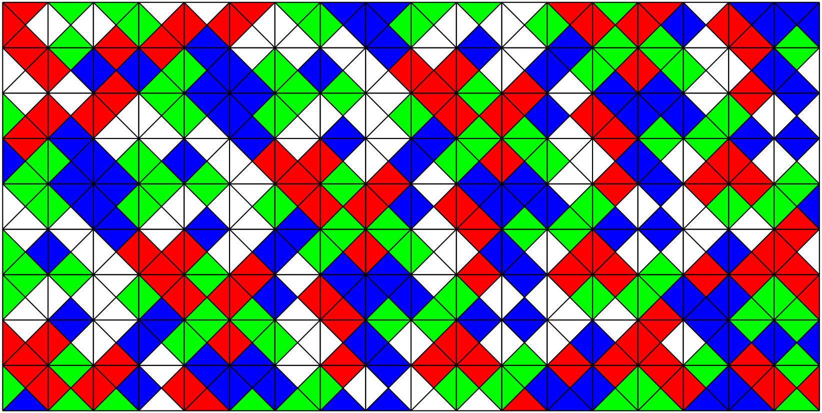


Given a set of Wang tiles, one try to tile the plane with copies of tiles in the set s.t. two adjacent sides have the same color

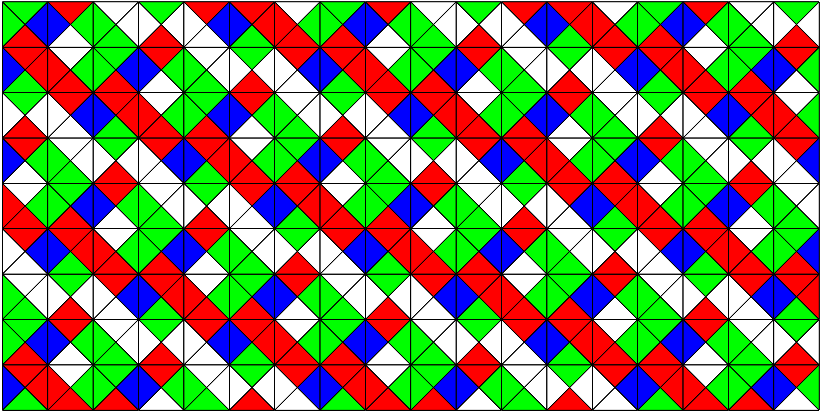


(No rotations !)

A tiling of the plane is *periodic* if there is a translation vector which does not change the tiling



A tile set is *periodic* if there is a periodic tiling of the plane with this set



A set is periodic if and only if there is a tiling with 2 (not colinear) translation vectors

A set is *finite* if there is no tiling of the plane with this set

A set is *aperiodic* if it tiles the plane, but no tiling is periodic

**Conjecture (Wang 1961)**

*Every set is either finite or periodic*

False:

**Theorem (Berger 1966)**

*It exists an aperiodic set of Wang tiles*

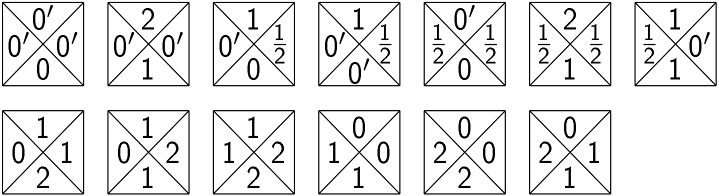
# History

- Berger : 20426 tiles in 1966 (lowered down later to 104)
- Knuth : 92 tiles in 1968
- Robinson : 56 tiles in 1971
- Ammann : 16 tiles in 1971
- Grunbaum : 24 tiles in 1987
- Kari and Culik : 14 tiles, then 13 tiles in 1996
- Here : 11 tiles (the fewest possible)

# “Kari-Culik” tile set

**Theorem (Kari-Culik 1996)**

*The following set (13 tiles) is aperiodic*



# New results

**Theorem**

*Every set with at most 10 Wang tiles is either finite or periodic*

**Theorem**

*There is a set with 11 Wang tiles which is aperiodic*

# Transducer

A set of Wang tiles can be seen as a transducer

A *transducer* is a finite automaton where each transition has an input letter and an output letter

$\mathcal{T} = (H, V, T)$ where $T \subseteq H^2 \times V^2$

We note $w\mathcal{T}w'$ if the transducer $\mathcal{T}$ writes $w'$ when it reads $w$

(Transducer on $\Sigma$ = Automaton on $\Sigma^2$)

# "Kari-Culik" tile set

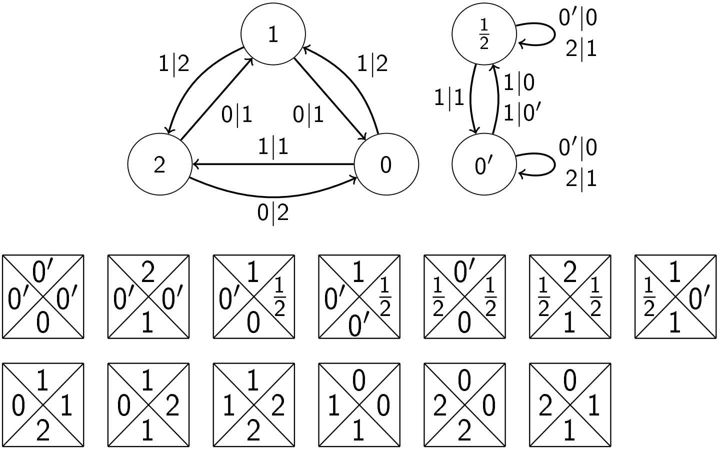



Figure: Kari-Culik tile set.

# Simplification

If $\mathcal{T}$ has a transition $a$ between two strongly connected components, then $\mathcal{T}$ is finite (resp. periodic) if and only if $\mathcal{T} \setminus \{a\}$ is finite (resp. periodic)

Let $s(\mathcal{T})$ be the union of strongly connected components of $\mathcal{T}$

$\mathcal{T}$ is finite (resp. periodic, aperiodic) if and only if $s(\mathcal{T})$ is finite (resp. periodic, aperiodic)

# Composition and power

Let $\mathcal{T} = (H, V, T)$ and $\mathcal{T}' = (H', V, T)$ be two transducers

Then $\mathcal{T} \circ \mathcal{T}' = (H \times H', V, T'')$ where:

$$T'' = \{((w, w'), (e, e'), s, n') : (w, e, s, x) \in T, (w', e', x, n')\}$$

$$\mathcal{T}^k = \mathcal{T}^{k-1} \circ \mathcal{T}$$

# Power

**Proposition**

*There is $k \in \mathbb{N}$ s.t. $s(\mathcal{T}^k)$ is empty iff $\mathcal{T}$ is finite*

**Proposition**

*There is $k \in \mathbb{N}$ s.t. there is a bi-infinite word $w$ such that $w\mathcal{T}^k w$ iff $\mathcal{T}$ is periodic*

# Enumeration (I)

To enumerate all sets with $n$ tiles, we compute all oriented graphs with $n$ arrows (with loops and multiple arrows)

For every pair of graphs $G$ and $G'$, we try every $n!$ bijections between the arrows of $G$ and $G'$

We only consider graphs without arrows between two strongly connected components.

| n | nb. graphs |
|---|---|
| 8 | 2518 |
| 9 | 13277 |
| 10 | 77810 |
| 11 | 493787 |

# Enumeration (II)

For every generated set $\mathcal{T}$, we compute $s(\mathcal{T}^k)$ until:

- $s(\mathcal{T}^k)$ is empty $\rightarrow$ the set is finite
- $\exists w$ s.t. $ws(\mathcal{T}^k)w$ $\rightarrow$ is periodic
- The computer run out of memory $\rightarrow$ the computer cannot conclude...

Optimizations :

- Cut branches in the exploration of $n!$ bijections
- Make tests on $\mathcal{T}$ and $\mathcal{T}^{\mathrm{tr}}$ on the same time
- Use (sometimes) bi-simulation to simplify transducers

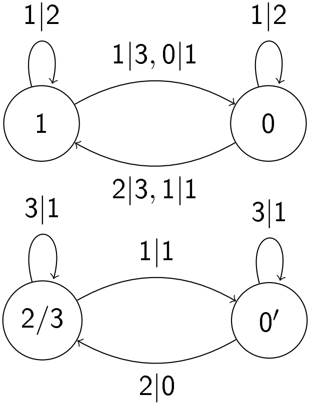
# Result ($n \leq 10$)

**Theorem**

*Every set of n Wang tile, $n \leq 10$, is finite or periodic*

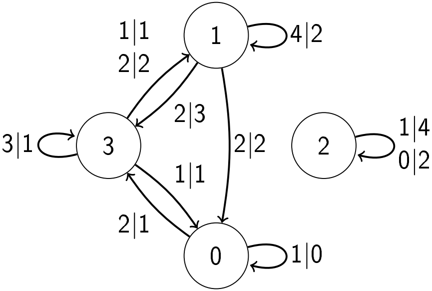
- $\sim 4$ days on $\sim 100$ cores
- Only one problematic case

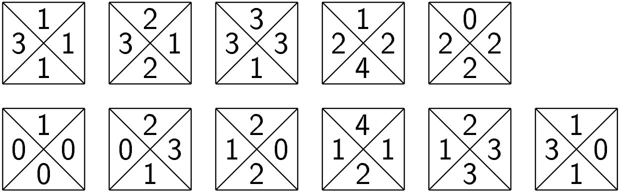
# The only problematic case with 10 tiles




“Kari-Culik” type $\times 2$, $\times\frac{1}{3}$. We have to use compactness to show that this tile-set is finite

# Aperiodic set of 11 tiles



# Proof (squetch.)

## Theorem

*$\mathcal{T}$ is aperiodic*

Ideas:

- It is the union if two transducers $\mathcal{T}_0$ and $\mathcal{T}_1$ (as for Kari-Culik)
- In a tiling by $\mathcal{T}$, we can merge layers into $\mathcal{T}_{10000}$ and $\mathcal{T}_{1000}$
- We get a new transducer $\mathcal{T}_D$ with 28 transitions, which is the union of $\mathcal{T}_a \simeq \mathcal{T}_{10000}$ and $\mathcal{T}_b \simeq \mathcal{T}_{1000}$
- We define the family $T_n$, with $T_{n+3} = T_{n+1} \circ T_n \circ T_{n+1}$
- We show $\mathcal{T}_b = T_0$, $\mathcal{T}_{aa} = T_1$, $\mathcal{T}_{bab} = T_2$
- The only admissible vertical word for $\mathcal{T}_D$ is the Fibonacci word

$\mathcal{T}$ is the union of $\mathcal{T}_0$ and $\mathcal{T}_1$, with (resp.) 9 and 2 tiles
For $w \in \{0,1\}^* \setminus \{\epsilon\}$, let $\mathcal{T}_w = \mathcal{T}_{w[1]} \circ \mathcal{T}_{w[2]} \circ \dots \mathcal{T}_{w[|w|]}$

**Fact**

*$s(\mathcal{T}_{11})$, $s(\mathcal{T}_{101})$, $s(\mathcal{T}_{1001})$ and $s(\mathcal{T}_{00000})$ are empty*

If $t$ is a tiling by $\mathcal{T}$, then there exists a bi-infinite word $w \in \{1000, 10000\}^{\mathbb{Z}}$ s.t. $t(x,y) \in T(\mathcal{T}_{w[y]})$

Let $\mathcal{T}_A = s(\mathcal{T}_{1000} \cup \mathcal{T}_{10000})$

There is a bijection between tilings by $\mathcal{T}$ and tilings by $\mathcal{T}_A$

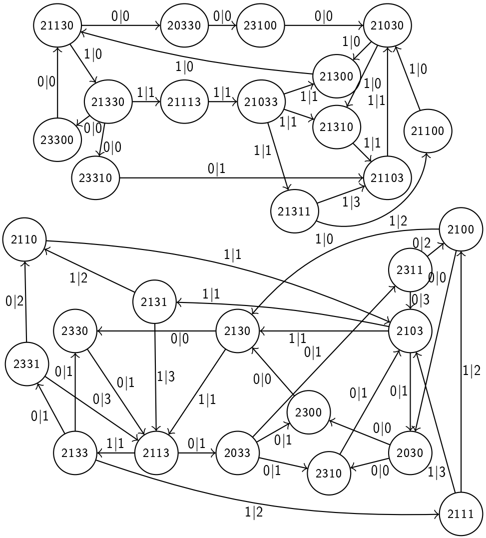



Figure: $\mathcal{T}_A$, the union of $s(\mathcal{T}_{10000})$ (top) and $s(\mathcal{T}_{1000})$ (bottom).

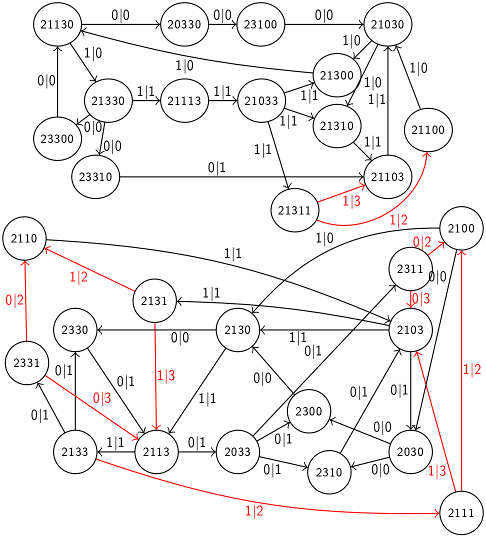



Figure: $\mathcal{T}_A$, the union of $s(\mathcal{T}_{10000})$ (top) and $s(\mathcal{T}_{1000})$ (bottom).

# Elimination of transitions with 2, 3 or 4

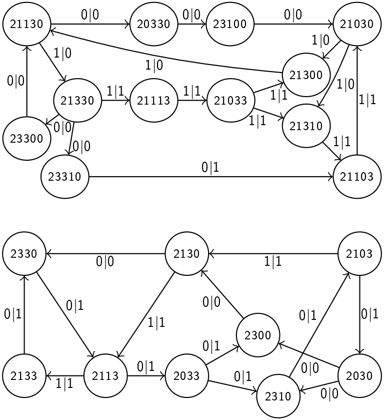



Figure: $\mathcal{T}_B = s(s(\mathcal{T}_A{}^{\mathrm{tr}})^{\mathrm{tr}})$.

# Simplification by bi-simulation

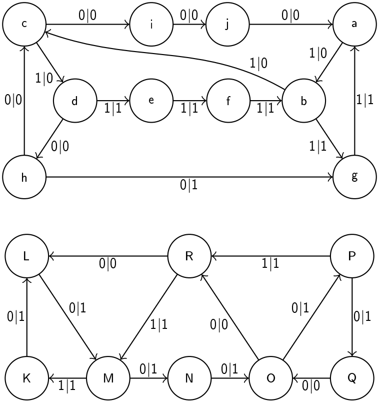



Figure: $\mathcal{T}_C$, "simplification" of $\mathcal{T}_B$.

## Proposition

*Let $(w_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence of bi-infinite words s.t. $w_i \mathcal{T}_C w_{i+1}$ for every $i \in \mathbb{Z}$.*
*Then for every $i \in \mathbb{Z}$, $w_i$ is $(010, 101)$-free*

This follow from the fact than $s((\mathcal{T}_C{}^{\mathrm{tr}})^3)$ does not contains the state 010, nor the state 101

Every tiling by $\mathcal{T}_C$ is in bijection with a tiling by $\mathcal{T}_D$

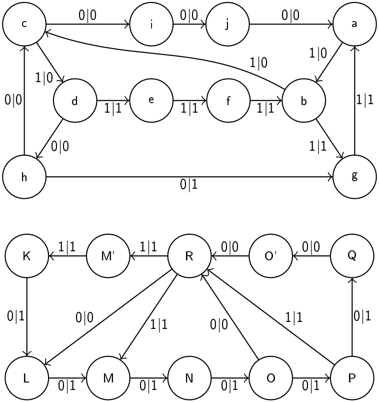
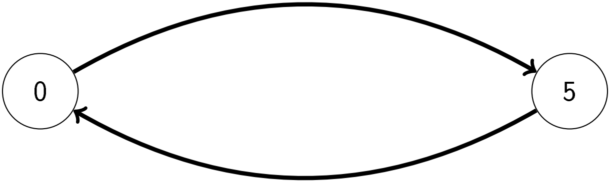



Figure: $\mathcal{T}_D$, the union of $\mathcal{T}_a$ (top) and $\mathcal{T}_b$ (bottom)

$T_n$ for even $n$:

$$\alpha : 0^{g(n+1)-3} | 1^{g(n+1)-3}$$

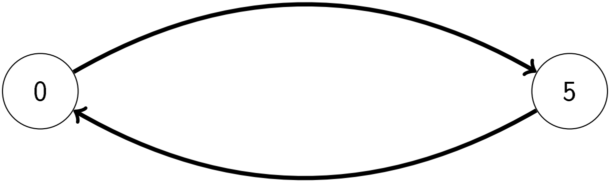



$$\begin{array}{ll}
\beta : 0^{g(n)+3} & |(100)1^{g(n)} \\
\gamma : 0^{g(n+2)+3} & |1^{g(n+1)}(000)1^{g(n)} \\
\delta : 0^{g(n)}(111)0^{g(n+1)} & |1^{g(n+2)+3} \\
\epsilon : 0^{g(n)}(110) & |1^{g(n)+3} \\
\omega : 0^{g(n+2)}(110)0^{g(n)} & |1^{g(n)}(100)1^{g(n+2)}
\end{array}$$

$g(n)$ is the $(n+2)$-th Fibonacci number: $g(0) = 2$, $g(1) = 3$, $g(n+2) = g(n+1) + g(n)$

$T_n$ for odd $n$:

$\mathbb{A} : 1^{g(n+1)-3} | 0^{g(n+1)-3}$

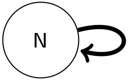



$$
\begin{array}{ll}
\mathbb{B} : 1^{g(n)+3} & |(110)0^{g(n)} \\
\mathbb{C} : 1^{g(n+2)+3} & |0^{g(n+1)}(111)0^{g(n)} \\
\mathbb{D} : 1^{g(n)}(000)1^{g(n+1)} & |0^{g(n+2)+3} \\
\mathbb{E} : 1^{g(n)}(100) & |0^{g(n)+3} \\
\mathbb{O} : 1^{g(n+2)}(100)1^{g(n)} & |0^{g(n)}(110)0^{g(n+2)}
\end{array}
$$

$g(n)$ is the $(n+2)$-th Fibonacci number: $g(0) = 2$, $g(1) = 3$, $g(n+2) = g(n+1) + g(n)$

# Case of $\mathcal{T}_b$

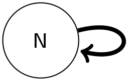
In $\mathcal{T}_b$, every long enough path passes through "N"
Thus $\mathcal{T}_b$ is equivalent to:




| | |
|---|---|
| 00000 | 10011 |
| 00000000 | 11100011 |
| 00111000 | 11111111 |
| 00110 | 11111 |
| 0000011000 | 1110011111 |
| 0010 | 1011 |
| 001000 | 111011 |
| 0000010 | 1110011 |
| 0011000 | 1011111 |

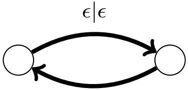
# Case of $\mathcal{T}_b$

In $\mathcal{T}_b$, every long enough path passes through “N”
Thus $\mathcal{T}_b$ is equivalent to:




$$\begin{array}{l|l}
00000 & 10011 \\
00000000 & 11100011 \\
00111000 & 11111111 \\
00110 & 11111 \\
0000011000 & 1110011111
\end{array}$$

$\mathcal{T}_b$ is equivalent to $T_0$.




$$\begin{array}{l|l}
0^5 & (100)1^2 \\
0^{5+3} & 1^3(000)1^2 \\
0^2(111)0^3 & 1^{5+3} \\
0^2(110) & 1^{2+3} \\
0^5(110)0^2 & 1^2(100)1^5
\end{array}$$

# Case of $\mathcal{T}_{aa}$

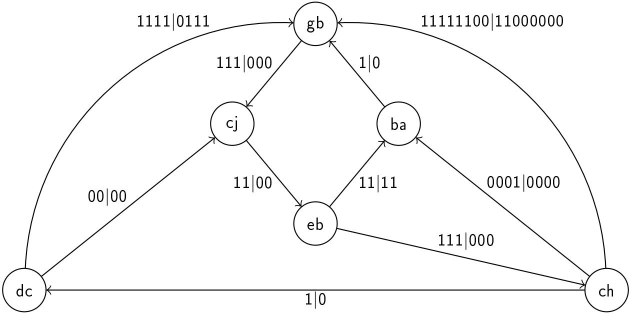
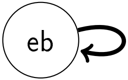



Figure: $s(\mathcal{T}_{aa})$

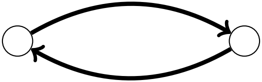
In $s(\mathcal{T}_{aa})$, every long path passes through “eb”. It is equivalent to:




11111111|11000000
1111111111111|0000011100000
1110001111111|0000000000000
11110011|00000000
111111110011111|000110000000000

$\mathcal{T}_{aa}$ is equivalent to $T_1$

$1^{5-3}|0^{5-3}$



$1^{3+3}|(110)0^3$
$1^{8+3}|0^5(111)0^3$
$1^3(000)1^5|0^{8+3)}$
$1^3(100)|0^{3+3}$
$1^8(100)1^3|0^3(110)0^8$

# Case of $\mathcal{T}_{bab}$

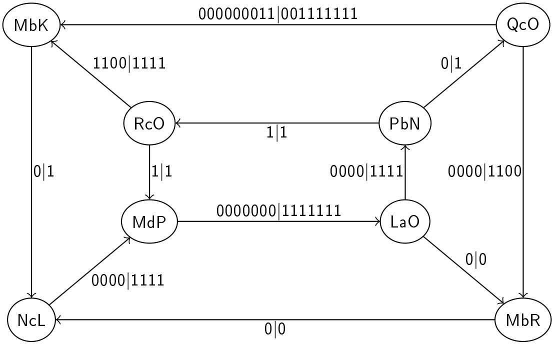
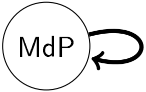


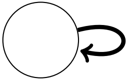

Figure: $s(\mathcal{T}_{bab})$

Every long path passes through "MdP". Thus it is equivalent to :




```
0000000000000             |1111111001111
000000000000000000000     |111111111111110001111
000000000001110000000     |111111111111111111111
0000000000011             |1111111111111
00000000000000000001100000|11111111111100111111111111
```

If we shift the input (3 times) and the output (6 times), we get:



```
0000000000000             |1001111111111
000000000000000000000     |111111110001111111111
000000001110000000000     |111111111111111111111
0000000011000             |1111111111111
00000000000000001100000000|11111100111111111111111111
```

$\mathcal{T}_{bab}$ is equivalent to $T_2$

**Fact**

*$s(\mathcal{T}_{bb})$, $s(\mathcal{T}_{aaa})$ and $s(\mathcal{T}_{babab})$ are empty*

If $t$ is a tiling by $\mathcal{T}_D$, then there is a bi-infinite word $w \in \{b, aa, bab\}^{\mathbb{Z}}$ s.t. $t(x,y) \in T(\mathcal{T}_{w[y]})$

That is, the tilings with $\mathcal{T}_D$ are images of the tilings by

$$\mathcal{T}_b \cup \mathcal{T}_{aa} \cup \mathcal{T}_{bab} \simeq T_0 \cup T_1 \cup T_2.$$

## $T_n$ for even $n$:

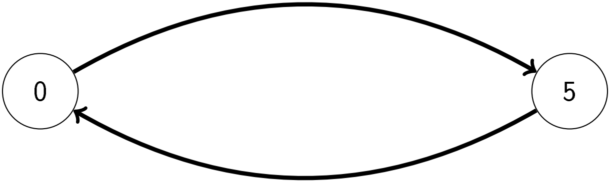
$$\begin{tikzcd}[column sep=huge]
0 \arrow[r, bend left, "\alpha : 0^{g(n+1)-3} | 1^{g(n+1)-3}"] & 5 \arrow[l, bend left]
\end{tikzcd}$$

$$\begin{array}{ll}
\beta : 0^{g(n)+3} & |(100)1^{g(n)} \\
\gamma : 0^{g(n+2)+3} & |1^{g(n+1)}(000)1^{g(n)} \\
\delta : 0^{g(n)}(111)0^{g(n+1)} & |1^{g(n+2)+3} \\
\epsilon : 0^{g(n)}(110) & |1^{g(n)+3} \\
\omega : 0^{g(n+2)}(110)0^{g(n)} & |1^{g(n)}(100)1^{g(n+2)}
\end{array}$$

$g(n)$ is the $(n+2)$-th Fibonacci number: $g(0) = 2$, $g(1) = 3$, $g(n+2) = g(n+1) + g(n)$

$T_n$ for odd $n$:

$$\mathbb{A} : 1^{g(n+1)-3} | 0^{g(n+1)-3}$$




$$\begin{array}{ll}
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\mathbb{D} : 1^{g(n)}(000)1^{g(n+1)} & |0^{g(n+2)+3} \\
\mathbb{E} : 1^{g(n)}(100) & |0^{g(n)+3} \\
\mathbb{O} : 1^{g(n+2)}(100)1^{g(n)} & |0^{g(n)}(110)0^{g(n+2)}
\end{array}$$

$g(n)$ is the $(n+2)$-th Fibonacci number: $g(0) = 2$, $g(1) = 3$, $g(n+2) = g(n+1) + g(n)$

One supposes than $n$ is even. (The odd case is similar.)

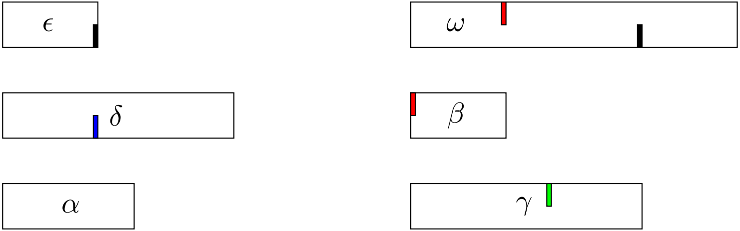
One supposes that $T_n \cup T_{n+1} \cup T_{n+2}$ tiles the plane

We show that $T_{n+1} \cup T_{n+2} \cup T_{n+3}$ tiles, and that:

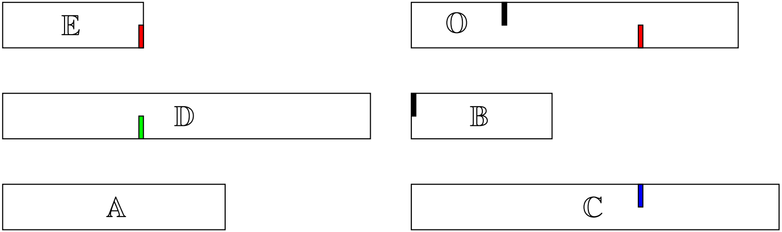
$$T_{n+3} \simeq T_{n+1} \circ T_n \circ T_{n+1}.$$

$T_n$ is surrounded by $T_{n+1}$
(output of $T_{n+1}$ has more 0's than 1's, $T_n$ and output $T_{n+2}$ have more 1's than 0's)

Transitions of $T_n$:




Transitions of $T_{n+1}$:



Let's take $T_n$. (We forget $\alpha$.)

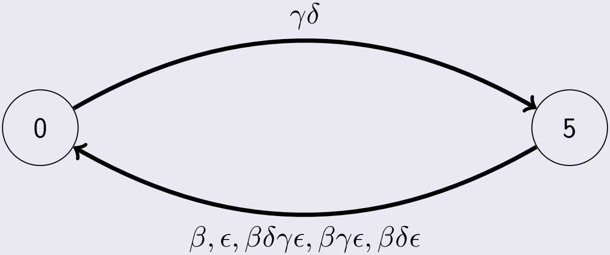
**Lemma**

*The following words cannot appear:*

- $\gamma\omega$,$\gamma\gamma$,$\gamma\beta$, $\beta\omega$, $\beta\beta$, $\beta\epsilon\beta$, $\gamma\epsilon\beta$, $\beta\delta\epsilon\beta$, $\gamma\delta\epsilon\beta$
- $\omega\delta$, $\delta\delta$, $\epsilon\delta$, $\omega\epsilon$, $\epsilon\epsilon$, $\epsilon\beta\epsilon$, $\epsilon\beta\delta$, $\epsilon\beta\gamma\epsilon$, $\epsilon\beta\gamma\delta$
- $\omega$

## Lemma

*Every infinite path in the transducer $T_n$ can be seen as an infinite path in the following transducer:*



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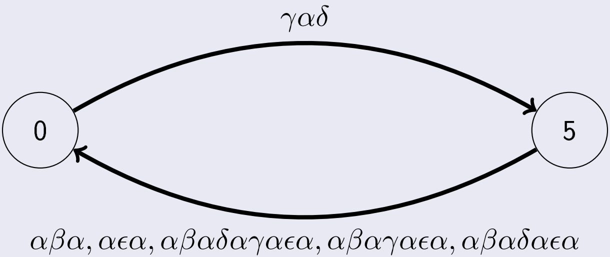
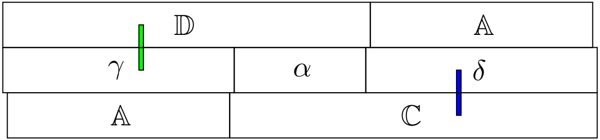
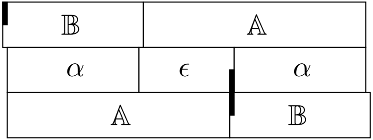


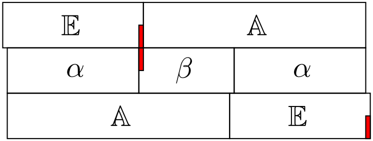
$\gamma\delta \to \mathbb{A}'$



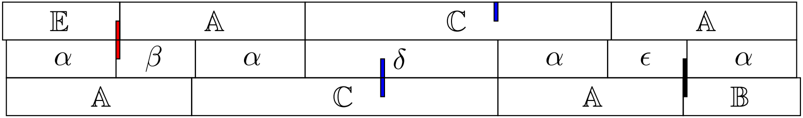

$\epsilon \to \mathbb{B}'$



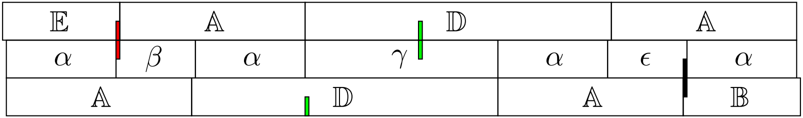
$\beta \to \mathbb{E}'$:



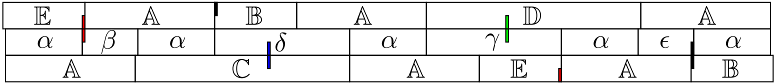

$\beta\gamma\epsilon \to \mathbb{C}'$:



$\beta\delta\epsilon \to \mathbb{D}'$:




$\beta\delta\gamma\epsilon \to \mathbb{O}'$:



It remains to show that one cannot have a stack of layers $T_{n+1}, T_n, T_{n+1}, T_n, T_{n+1}$

We can merge layers of a tiling with $T_n \cup T_{n+1} \cup T_{n+2}$ by:

1. $T_{n+1}$
2. $T_{n+2}$
3. $T_{n+1}\, T_n\, T_{n+1} \quad \simeq \quad T_{n+3}$

$T_n \simeq \mathcal{T}_{u_n}$ where:

$$u_0 = b, u_1 = aa, u_2 = bab$$

and

$$u_{n+3} = u_{n+1} u_n u_{n+1}$$

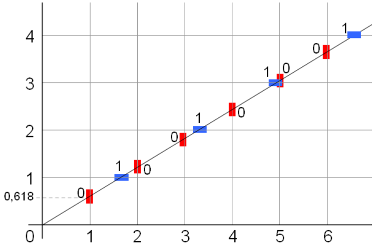
$$(u_n)_{n \geq 0} = (b, aa, bab, aabaa, babaabab, \ldots)$$

is a sequence of factors of the Fibonacci word
(the "singular factors")

# Fibonacci Word

Aperiodic word

$$w_f = abaababaabaababaababaabaababaabaababaababaa\ldots$$




Fixed point of the morphism $a \to ab$, $b \to a$

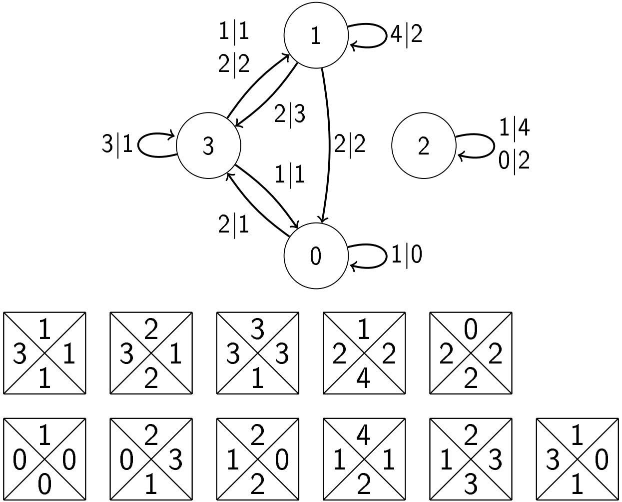
$v_0 = a$, $v_1 = ab$ and $v_{n+2} = v_{n+1}v_n$

$v_i$ converges to $w_f$

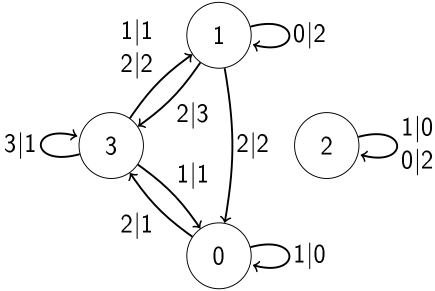
$(v_n)_{n\geq 0} = (a, \underline{a}b, \underline{ab}a, \underline{abaa}b, \underline{abaabab}a, \underline{abaababaabaa}b, \ldots)$

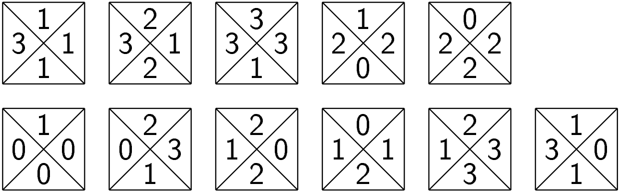
$(u_n)_{n\geq 0} = (b, a\underline{a}, b\underline{ab}, a\underline{abaa}, b\underline{abaabab}, a\underline{abaababaabaa}, \ldots)$

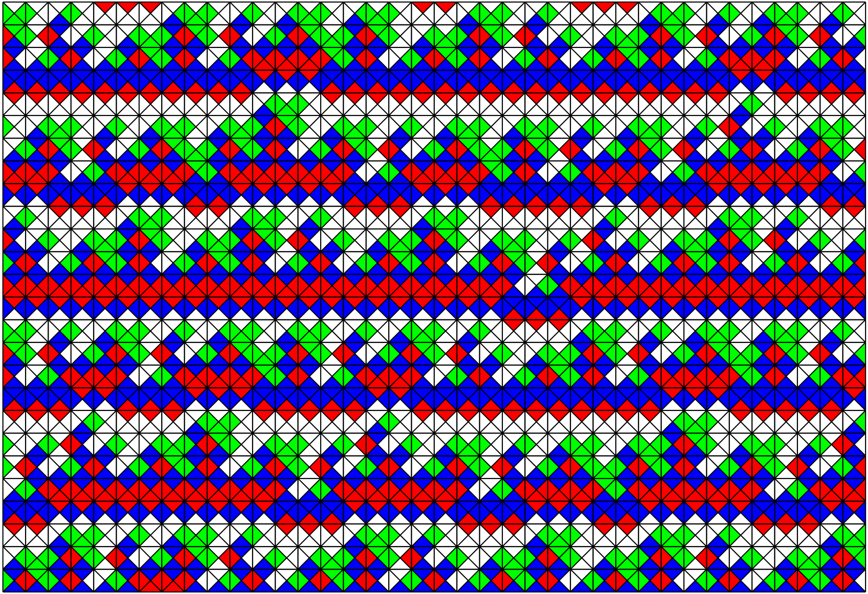
# Aperiodic set with 11 tiles



# Aperiodic set with 11 tiles and 4 colors



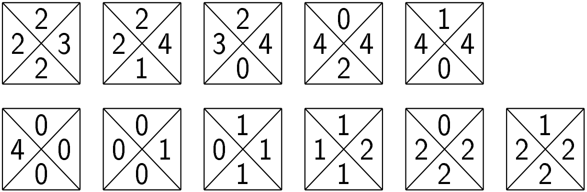



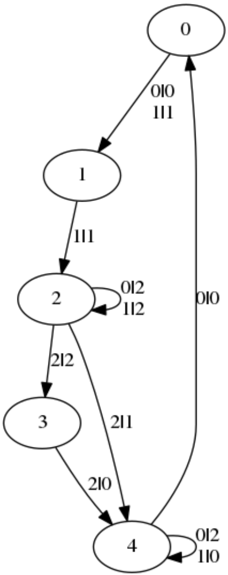


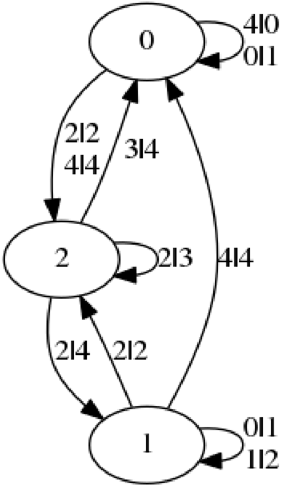
# Open question 1 : Another aperiodic set ?

Tiles sets with 11 tiles:

- 2 aperiodic (and 1 other probably very close)
- 23 others “candidates”
- 9 of “Kari-Culik” type (and probably finite)
- 14 not “Kari-Culik”
- 1 strange (interesting) candidate:




0
0|0
1|1
1
1|1
2
0|2
1|2
0|0
2|2
2|1
3
2|0
4
0|2
1|0


0
4|0
0|1
2|2
4|4
3|4
2
2|3
4|4
2|4
2|2
1
0|1
1|2

# Open question 2 : "proof from the book" ?

If we look at densities of 1 on each line on an infinite tiling, one transducer add $\varphi - 1$ and the other add $\varphi - 2$.
$\rightarrow$ "additive" Kari-Culik ?