Lots of Aperiodic Sets of Tiles

Chaim Goodman-Strauss
(some joint with Th. Fernique)

Lots of Aperiodic Sets of Tiles

(50625 + 9 to be exact)

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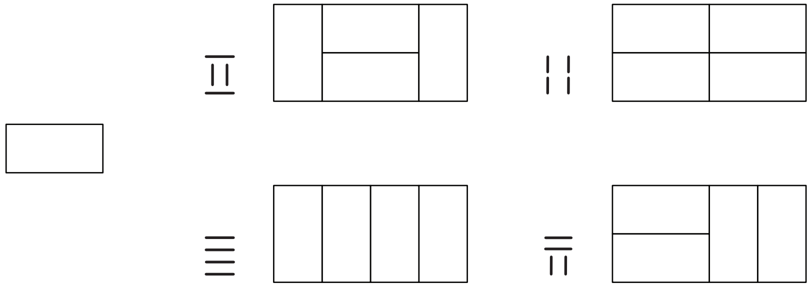
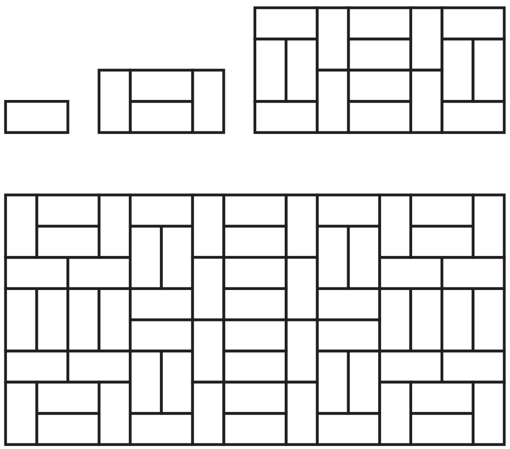
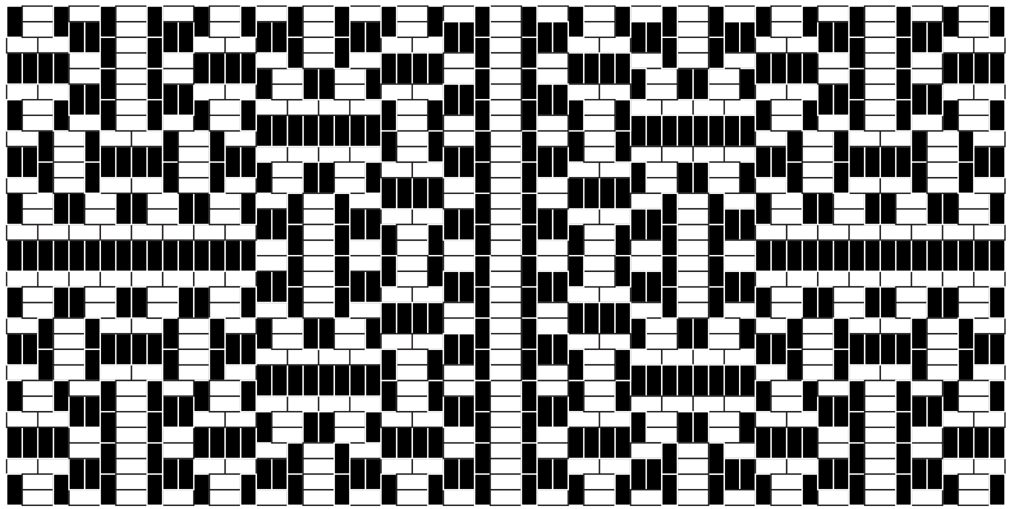
We will enforce

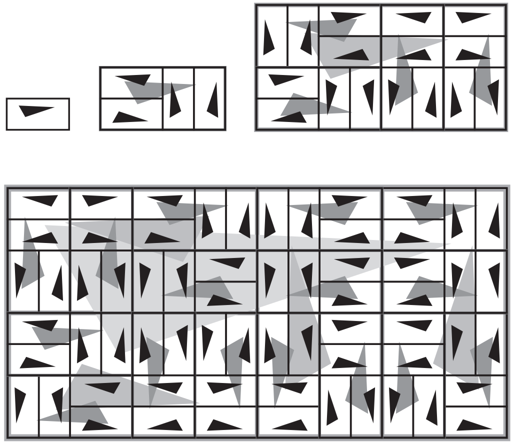


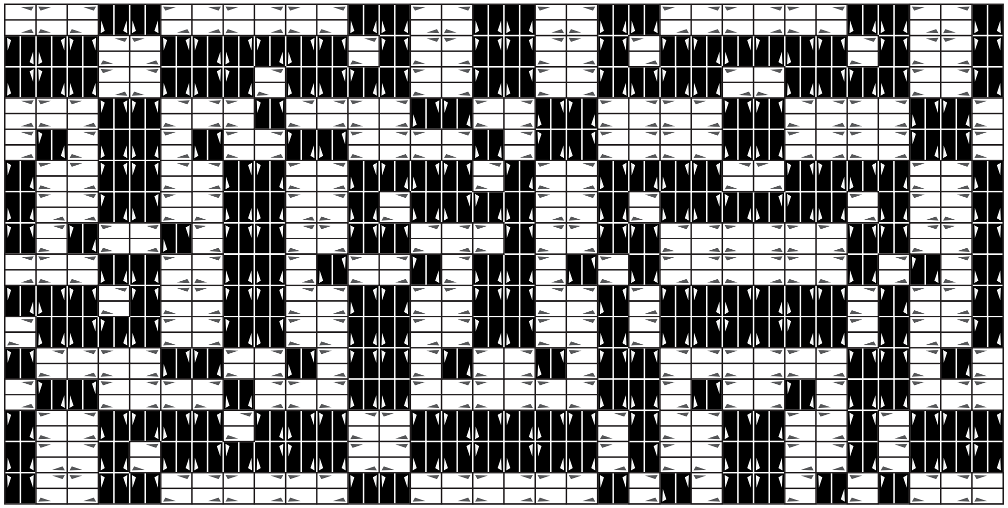
There are three unambiguous substitution rules for replacing a 2x1 rectangle by four smaller copies, shown at left.





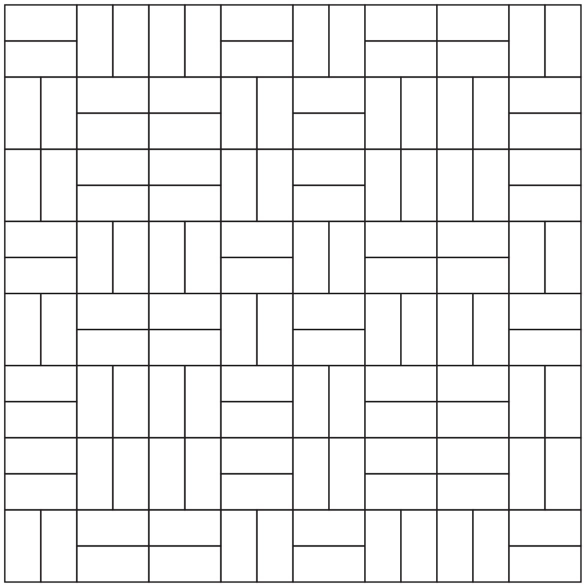
However the fourth rule is ambiguous when a second substitution is performed. We must know which end of the rectangle is which—and for each subsequent substitution we must further know right from left.

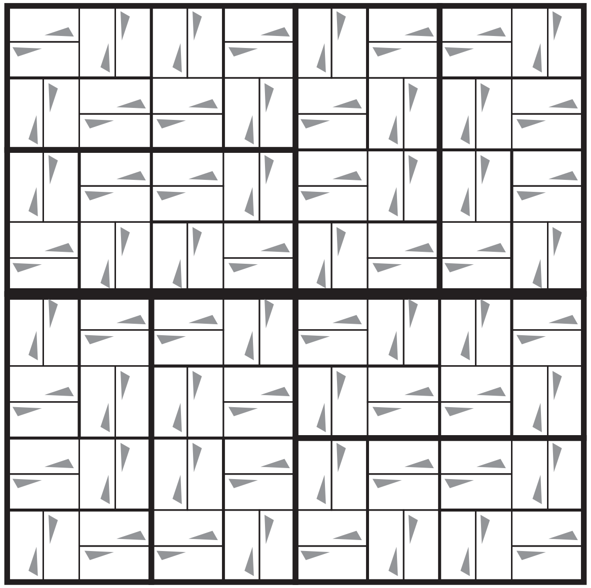


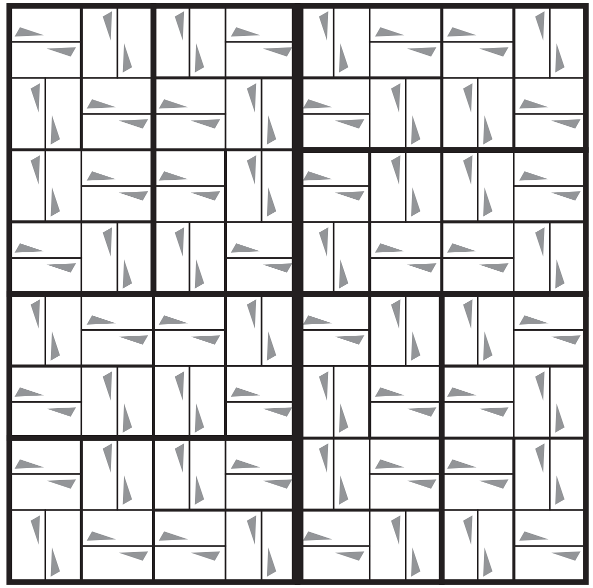


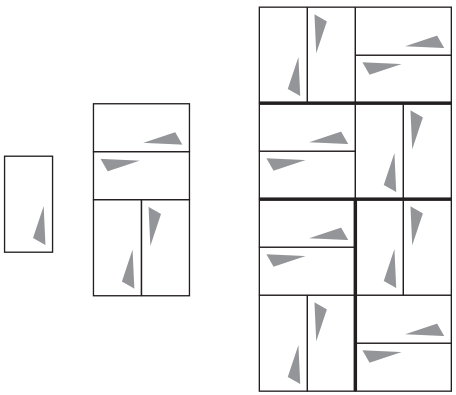
There are some interesting surprises:

Most notably, there are **non-periodic examples with non-unique decomposition**!

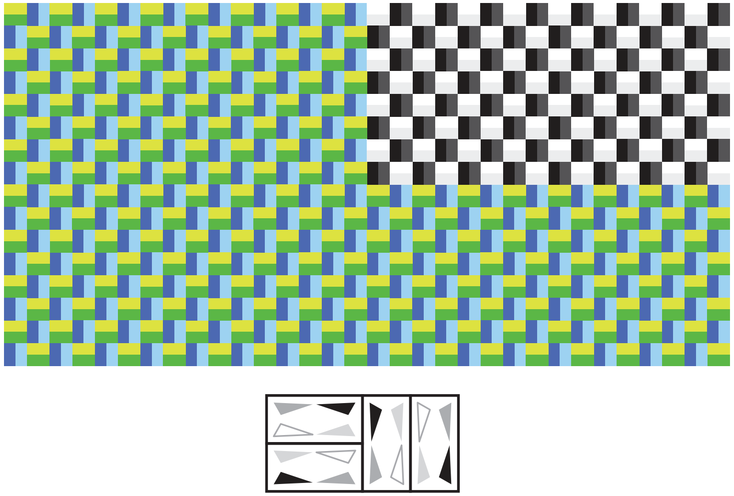


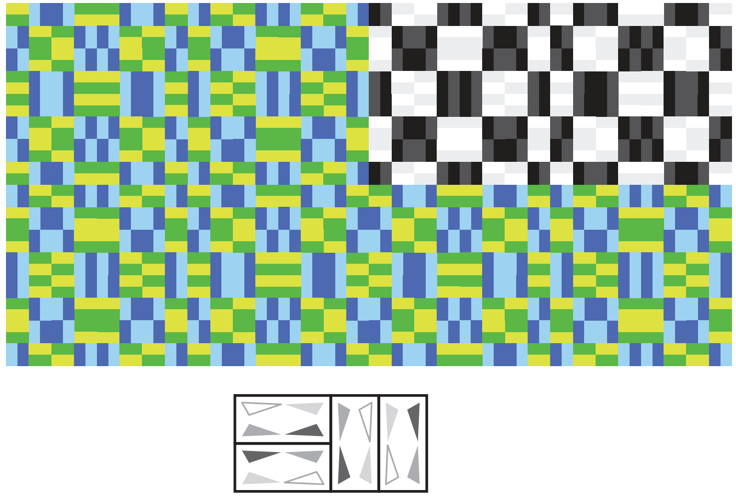


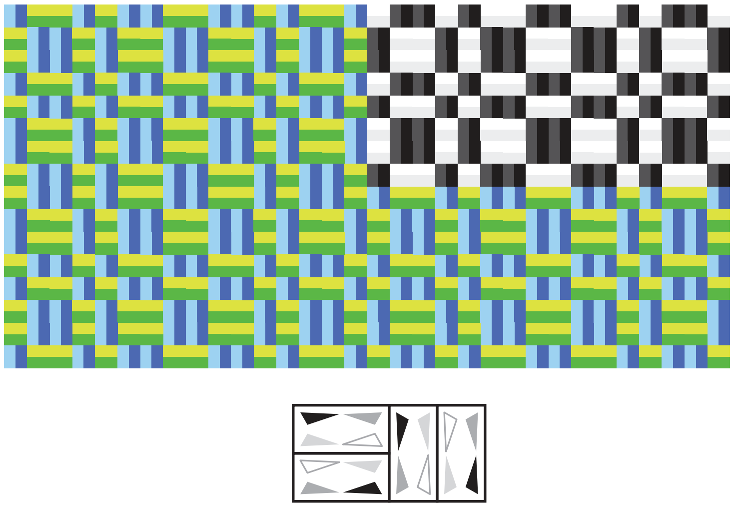


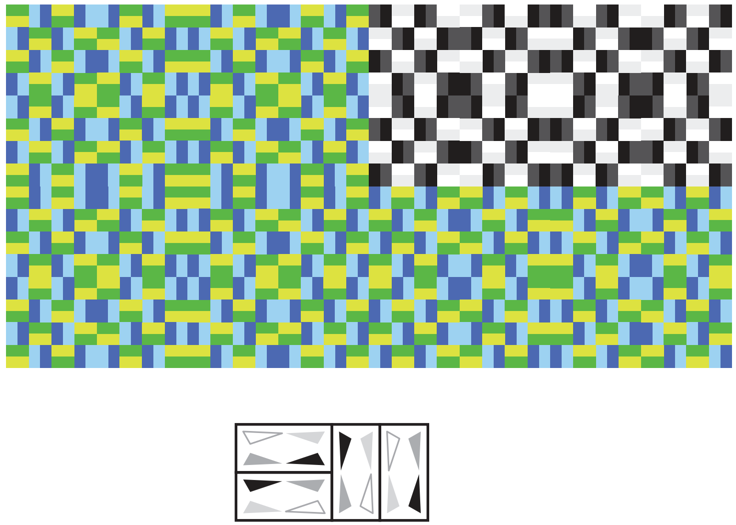


(There are 4 families of these with non-unique decomposition, three are non-periodic)

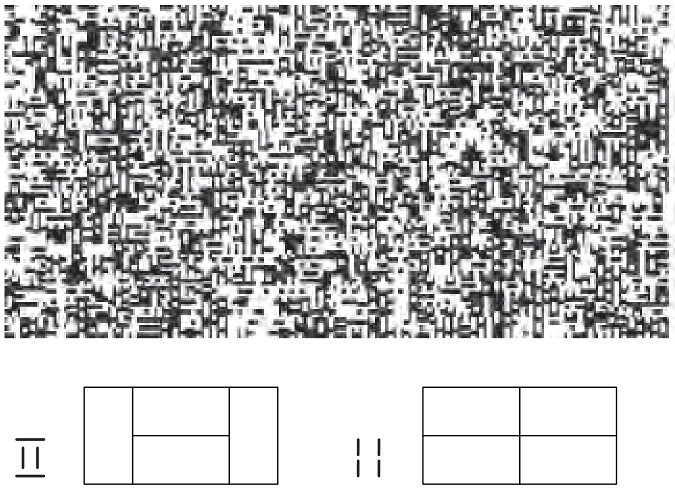


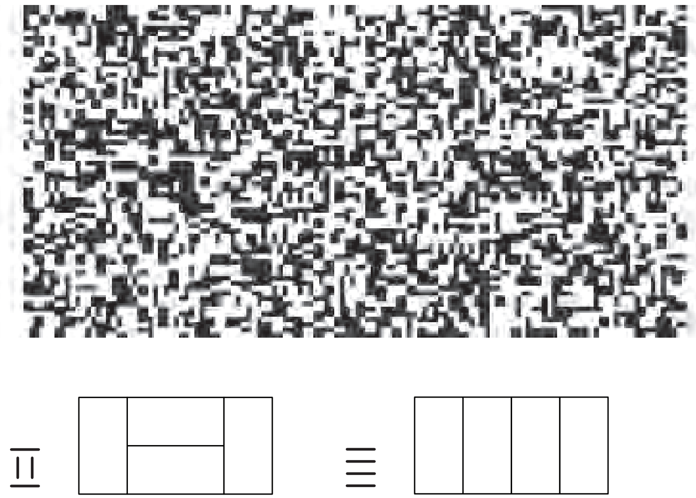






We will consider non-deterministic rules as well:
For example:





Each system has a natural entropy, measuring the order of the statistically typical substitution tiling, qualitatively evident in the two supertiles at left. We envision more elaborate structures built upon the tiles we present here for broader information theoretic applications.

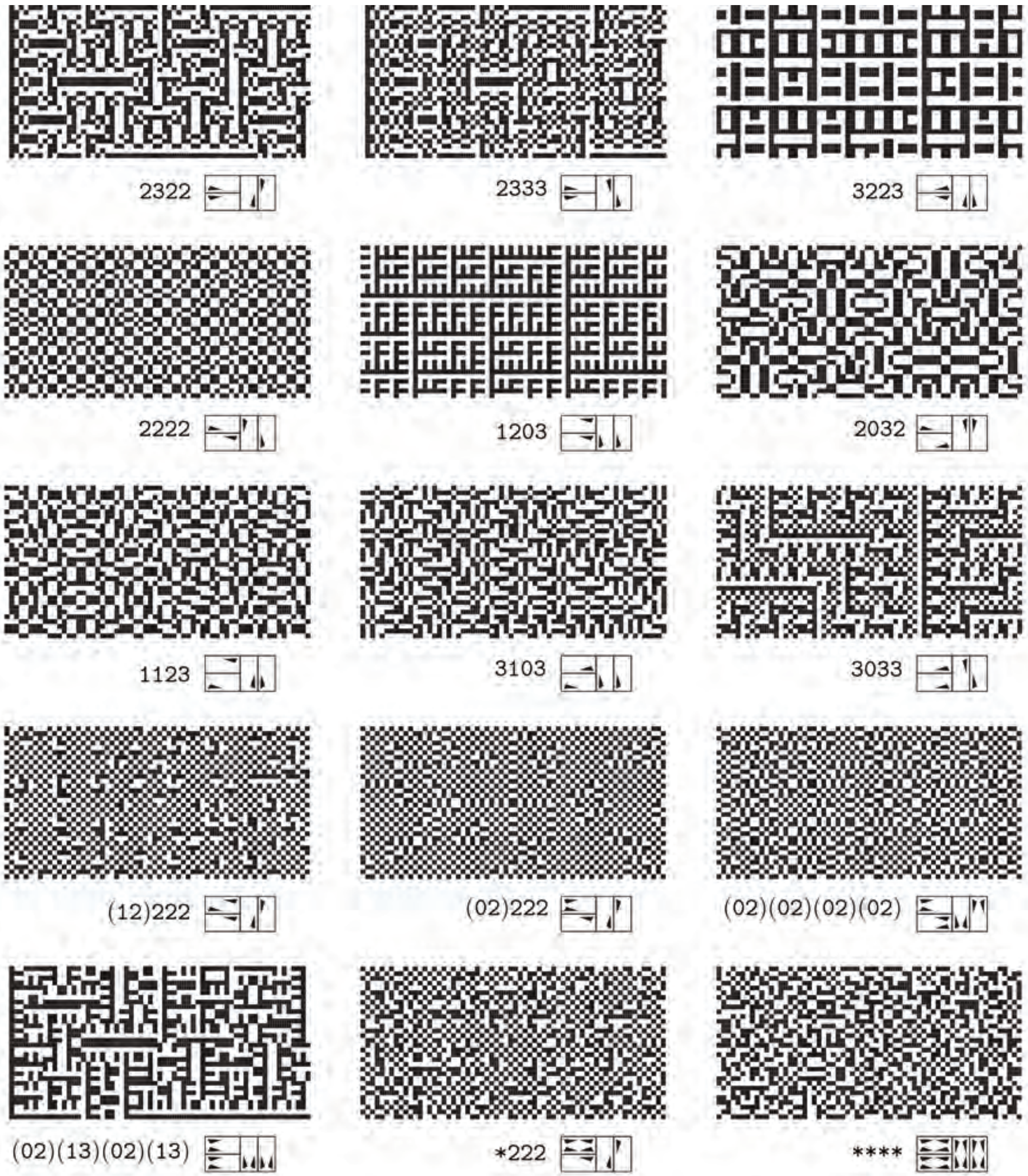


Figure 4: A few of the 50625 $\bar{\pi}$ tiling substitution rules, each specified by a code (Section 4) indicating the allowed orientation of each smaller child supertile in within its parent in the hierarchy

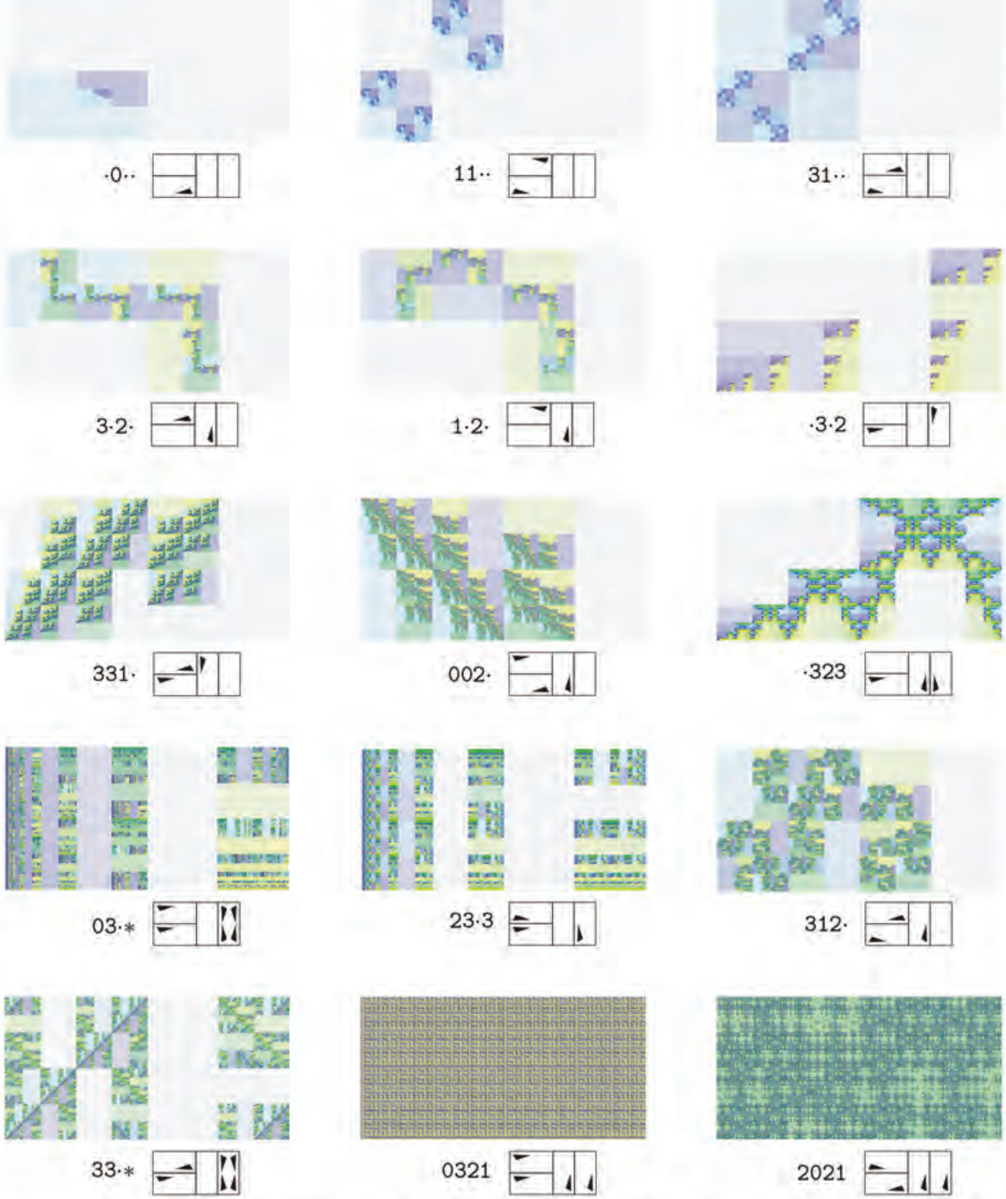
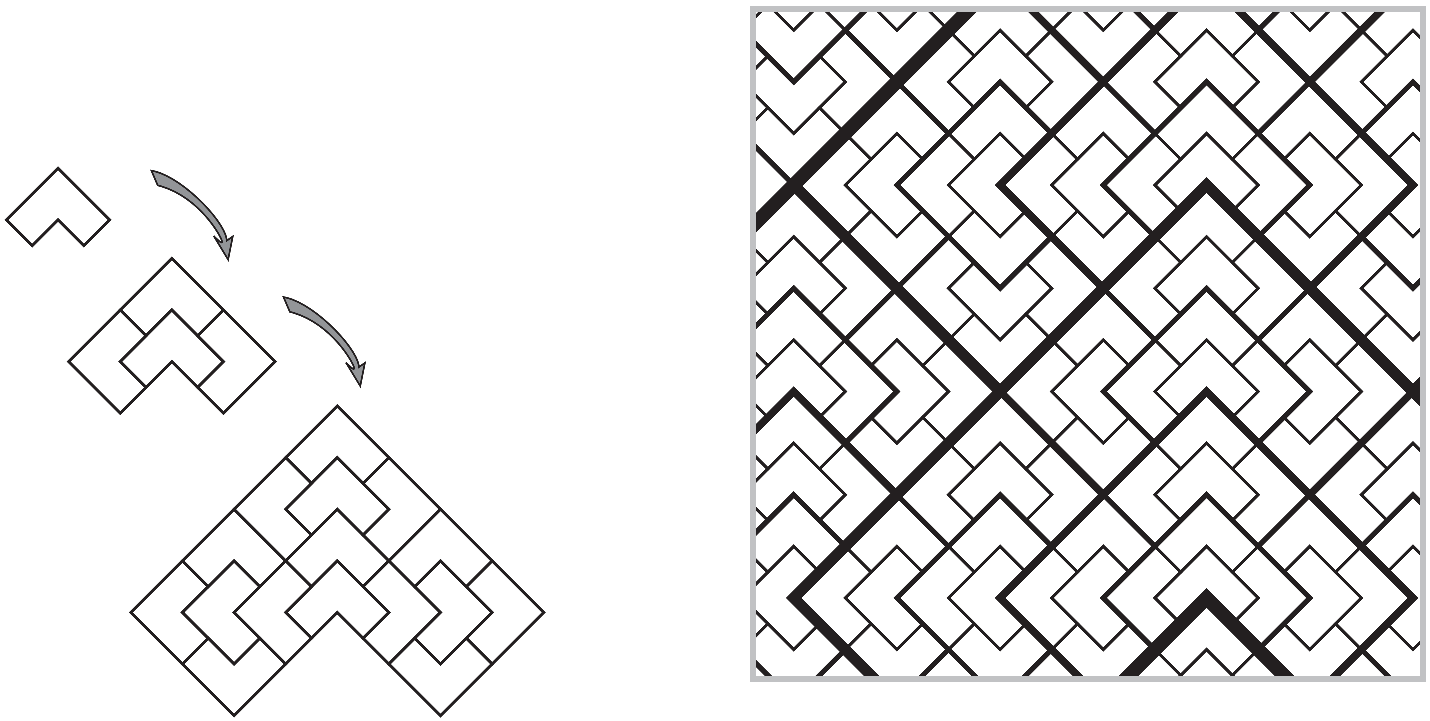
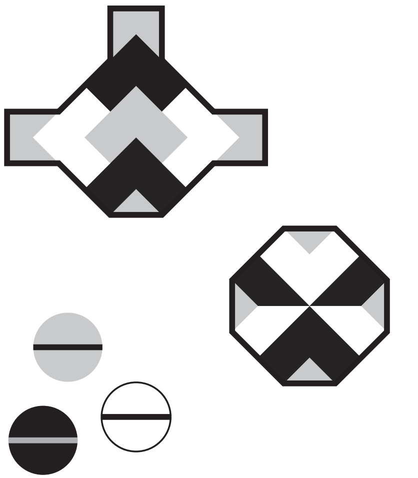


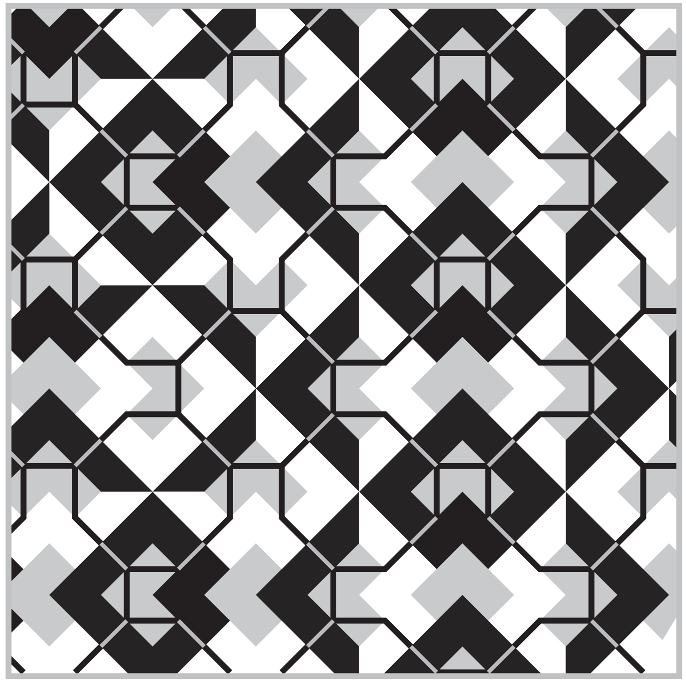
Figure 5: "Partial" $\overline{\pi}$ substitution rules, with empty digits denoted '·' indicating no tile is placed. The colors indicate the specific orientation of each tile in the substitution. At top left, an "elementary" rule, with just one non-empty digit. At top right and second row, "atomic" rules with two non-empty digits— these are the basis for assembling our aperiodic sets of matching rules. In the remaining rows, we show rules with fewer empty digits. By selectively highlighting markings on our final sets of tiles, we may pick out and enforce such structures, in a rich unexplored variety of forms, suitable as strata for other tiling tasks. On the bottom row, for comparison, two 8-level full supertiles scaled and colored in the same manner as these partial ones.

A substitution tiling



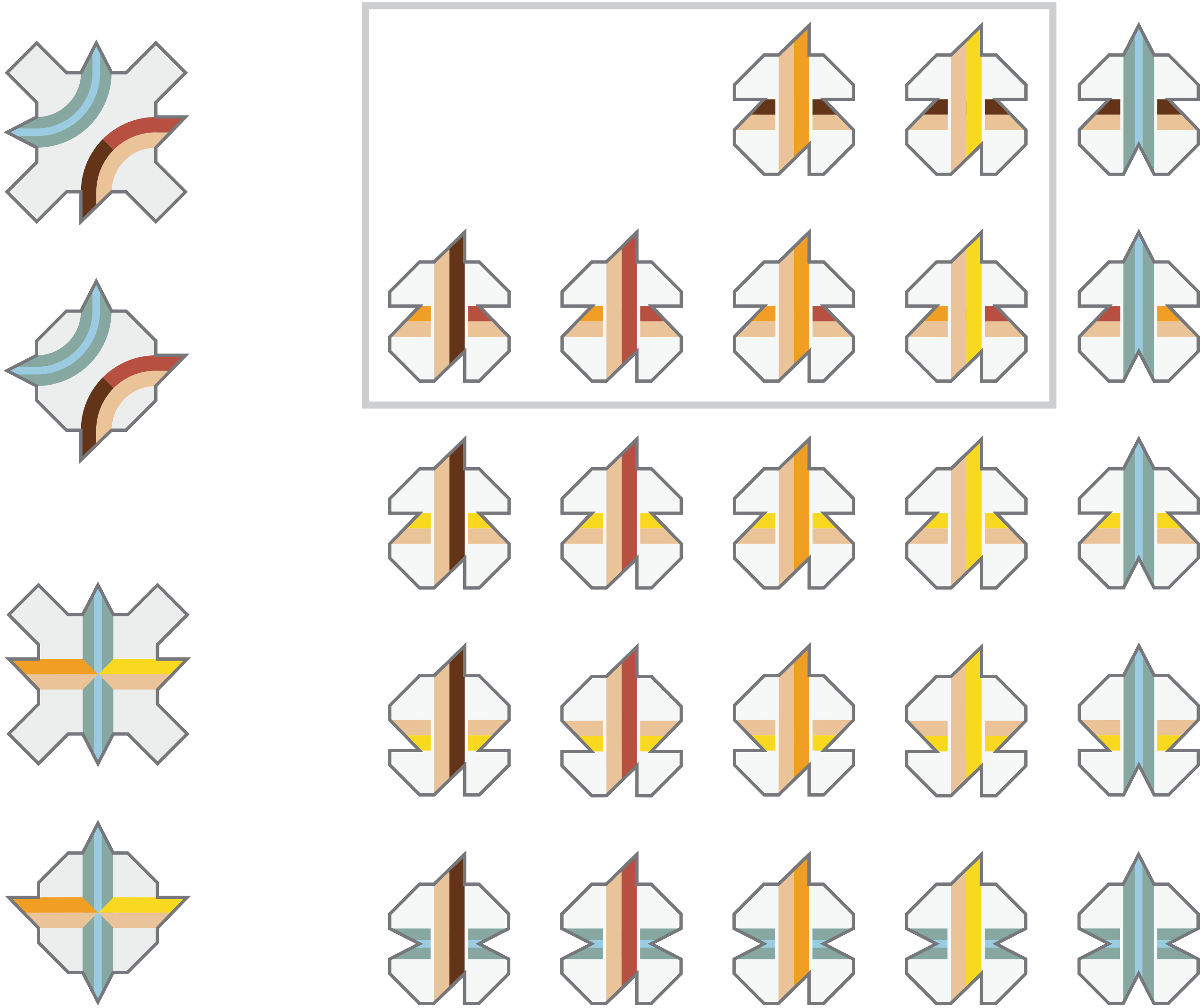
A matching rule tiling *enforcing* the substitution.

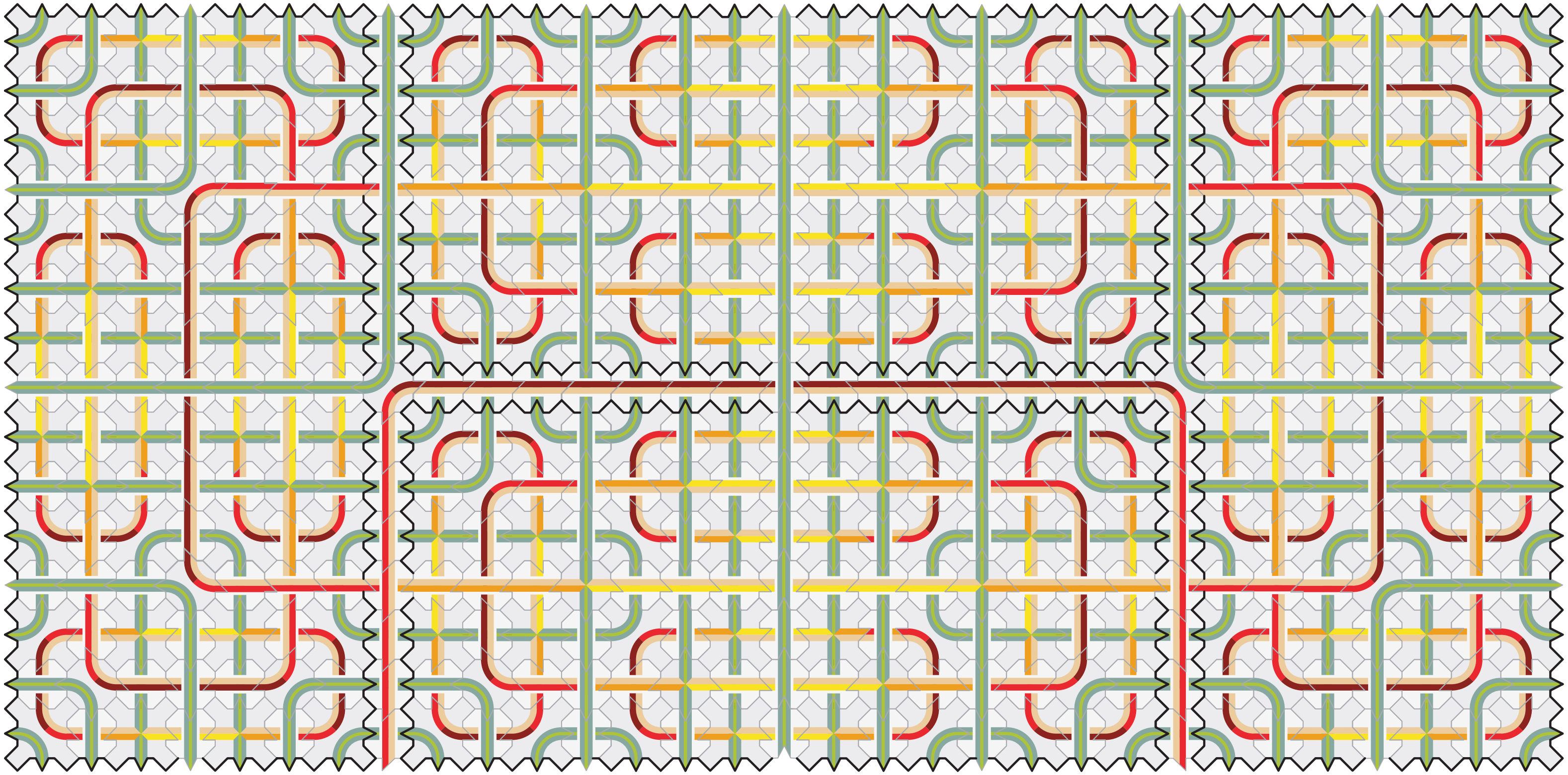


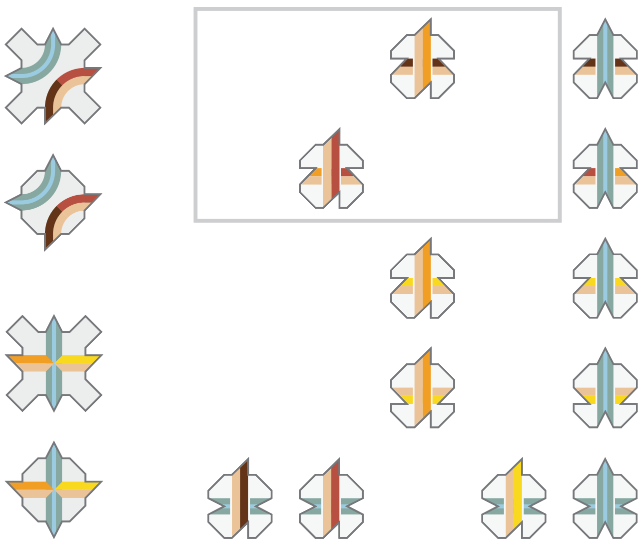


(trilobite and crab, GS)

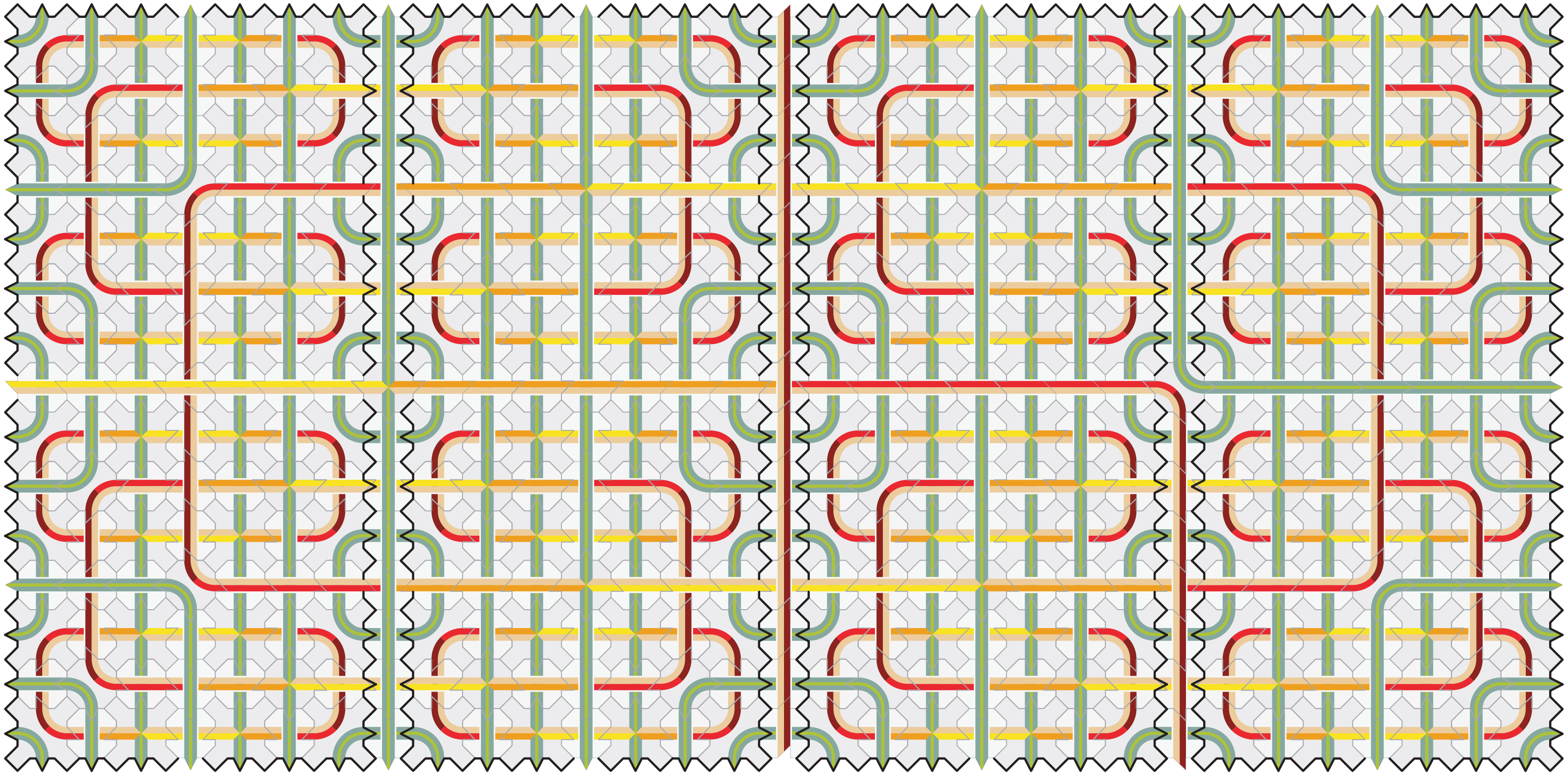
Tile set one: 9 aperiodic subsets of 27 tiles, enforcing various table substitutions.

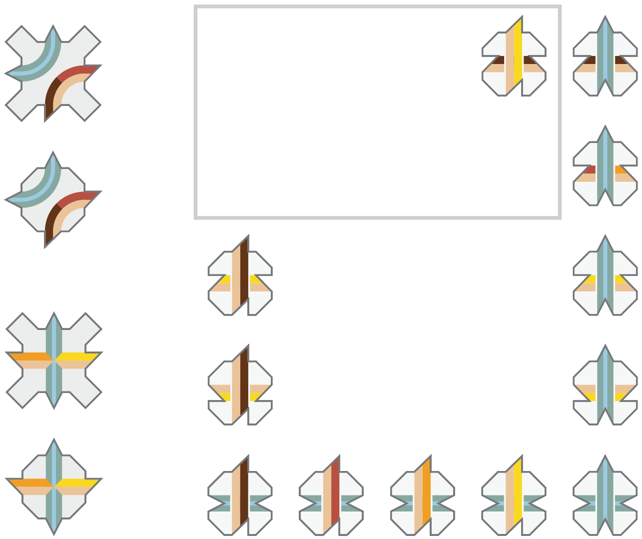


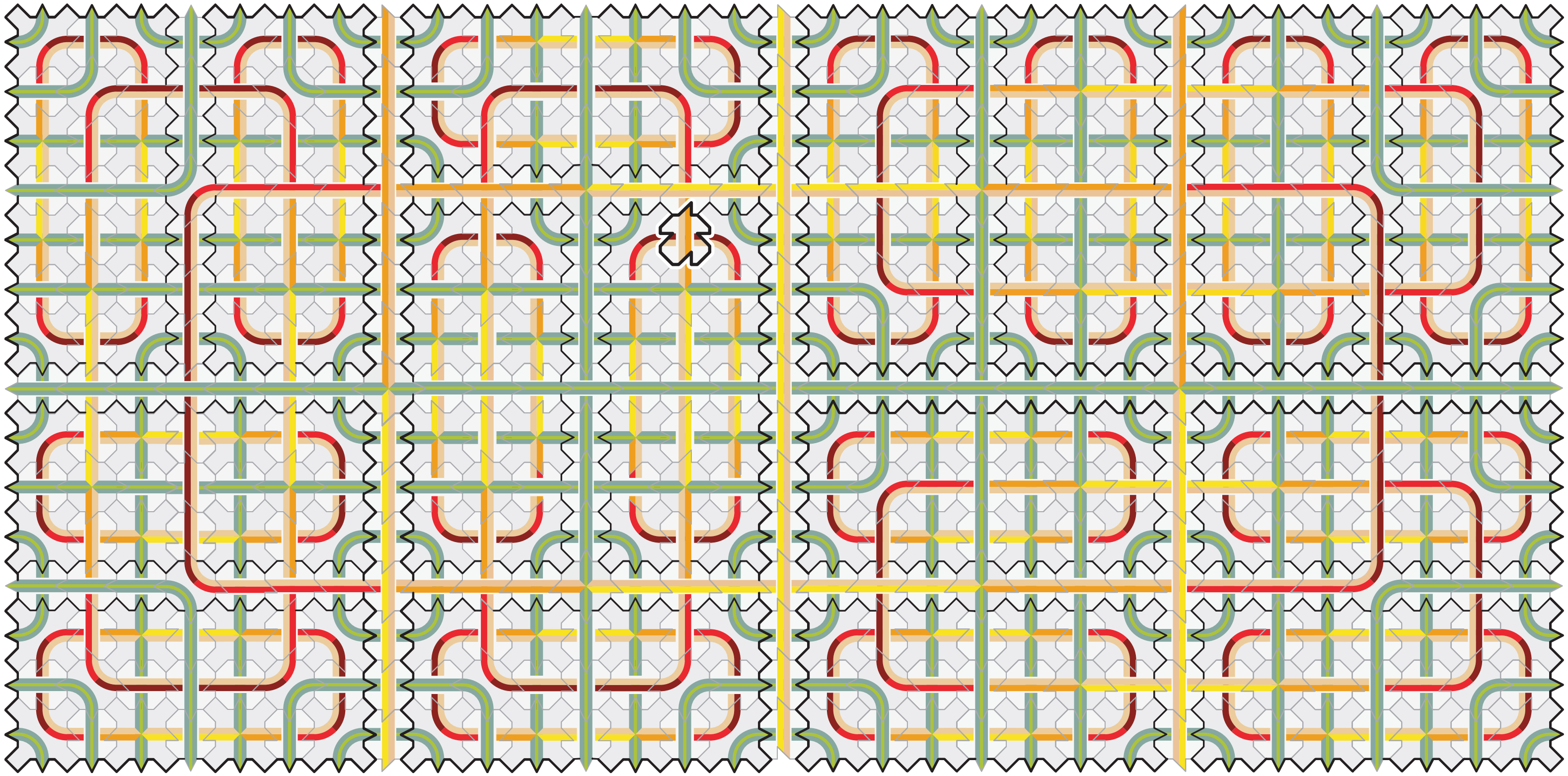


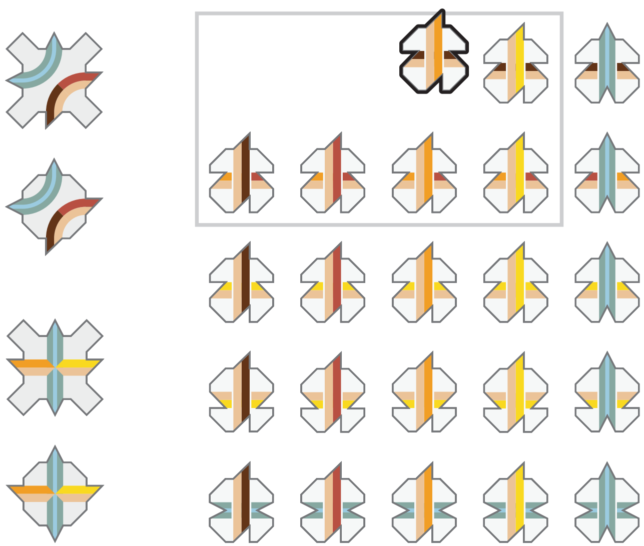


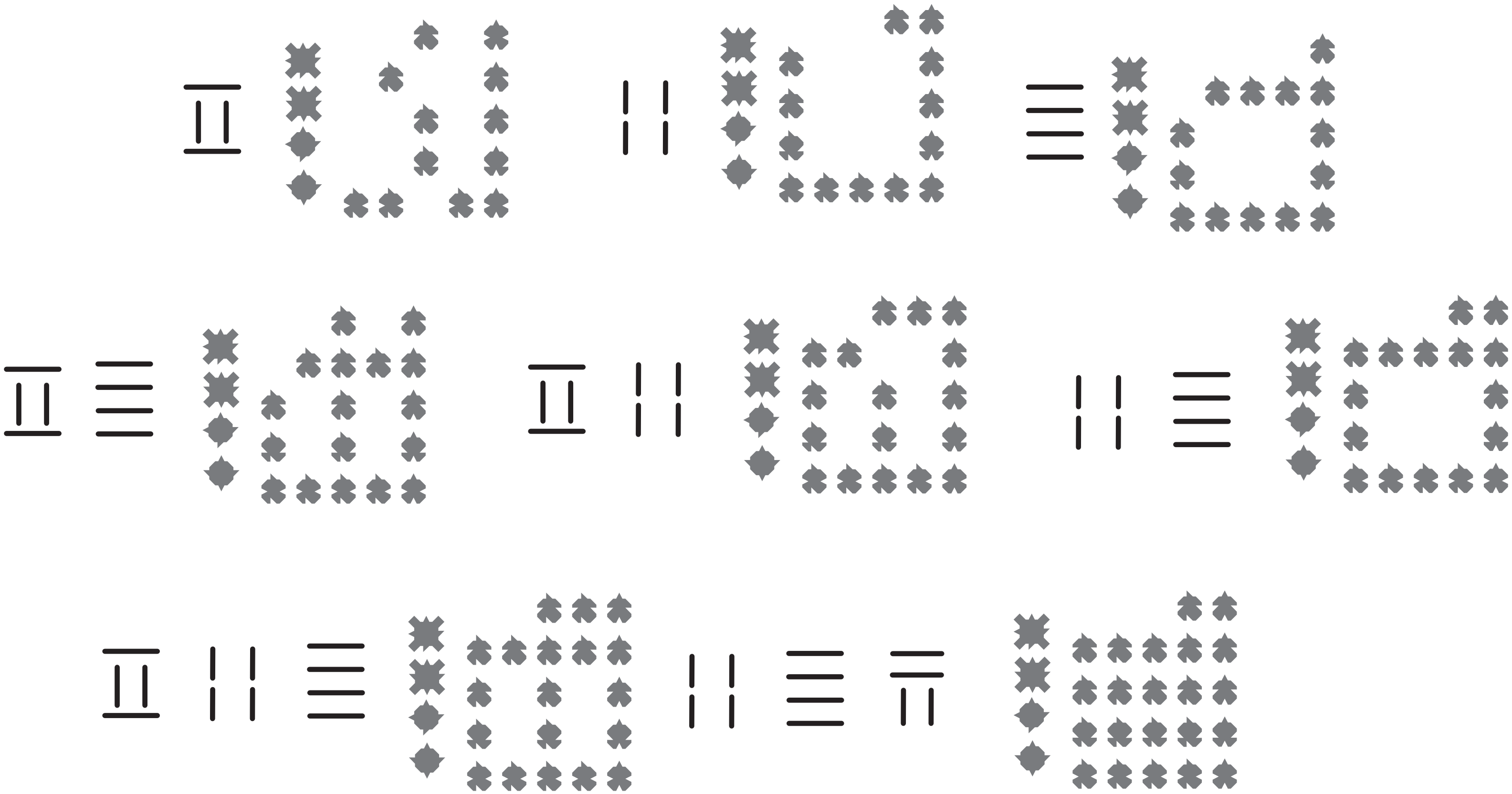
Joint with Thomas Fernique



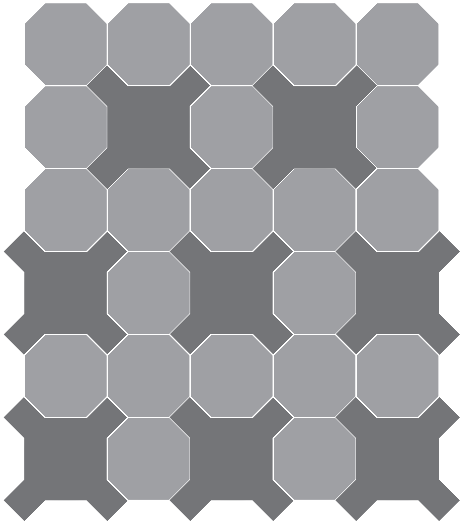


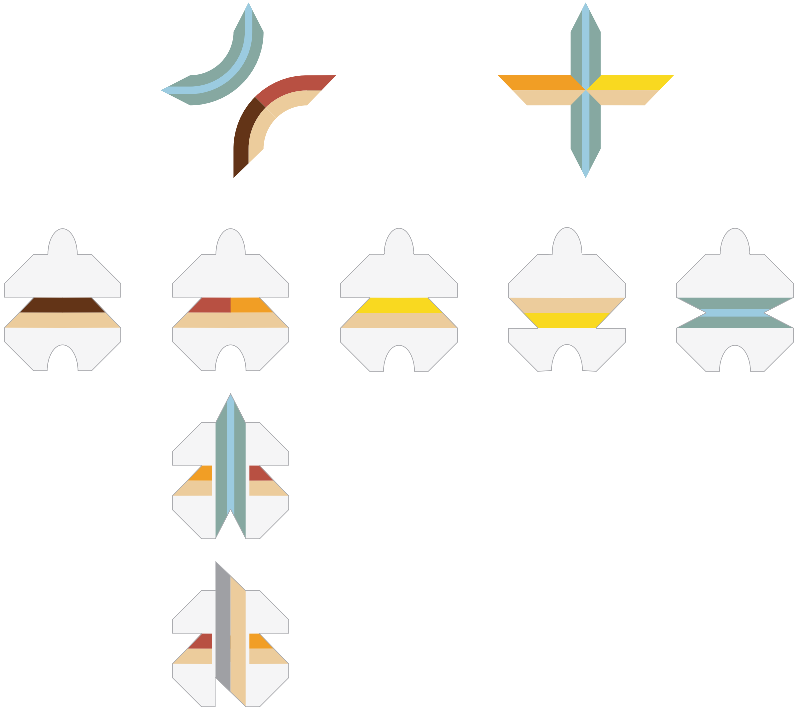


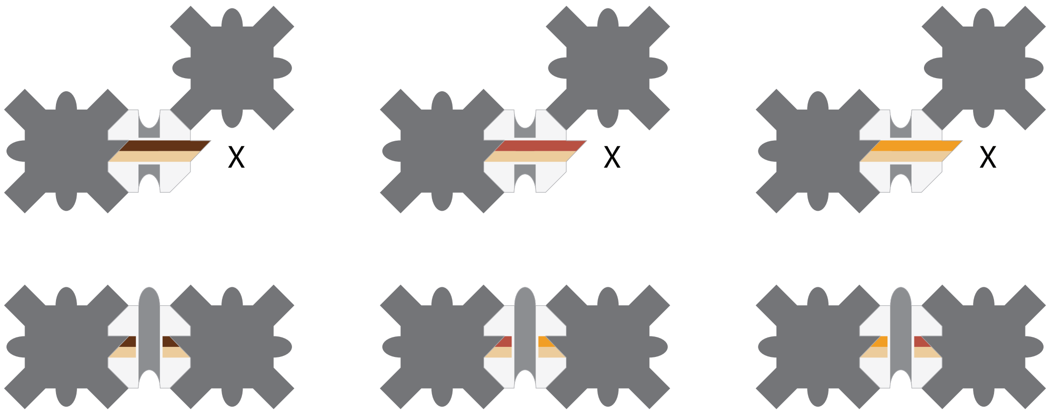


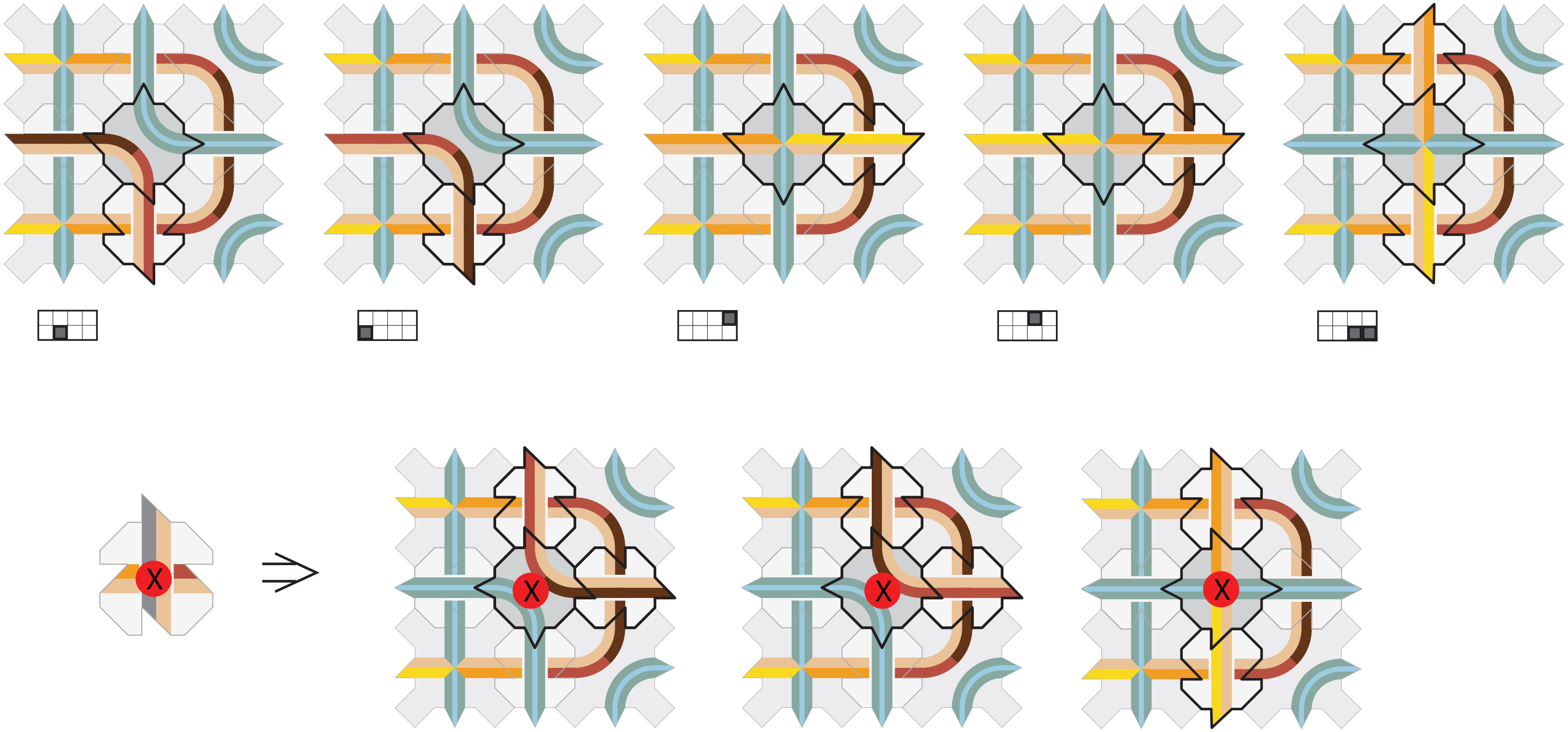


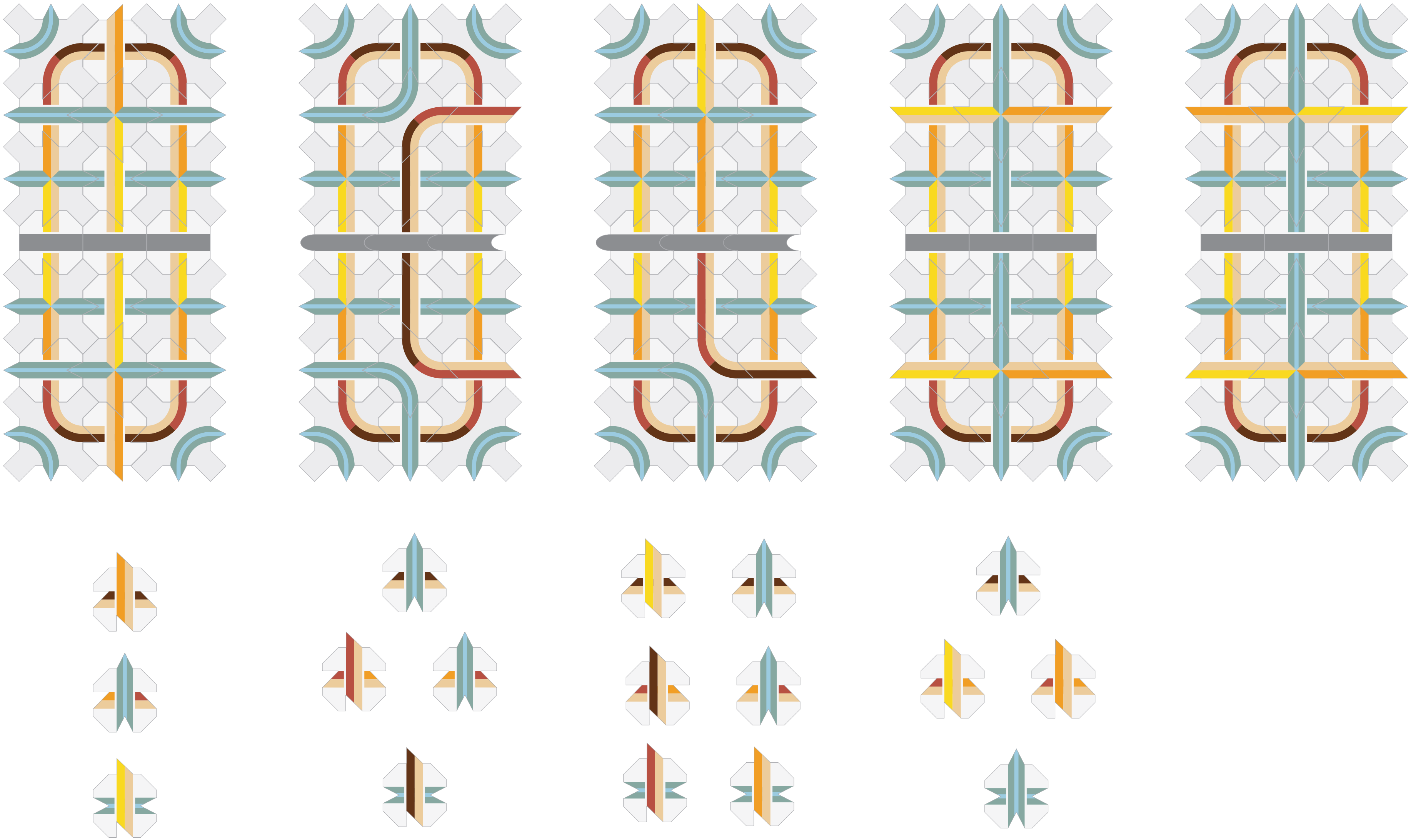
9 aperiodic sets from
27 tiles,
forcing table tilings

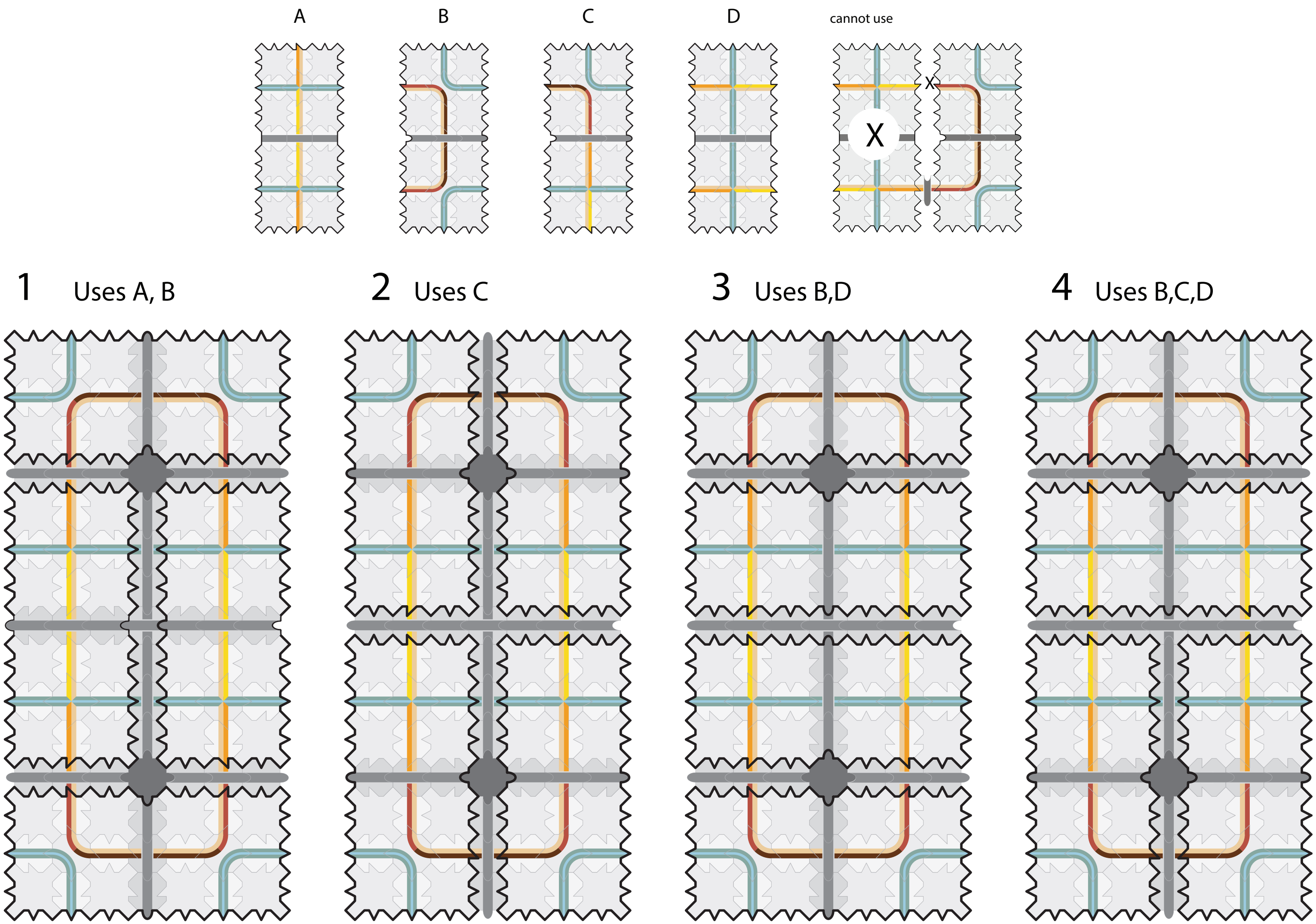





X
X
X


X
X
X
X



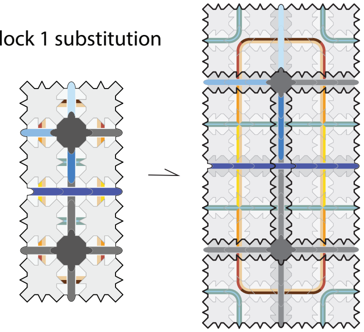


We have the following possible subsets:

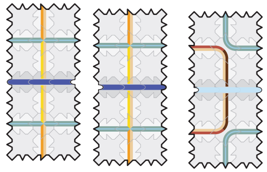
C (block 2, and a deterministic substitution)
A, B (block 1 and a deterministic substitution)
A, D (block 3 and a deterministic substitution)
A, B, C (blocks 1 and 2)
A, B, D (blocks 1 and 3)
A,C,D (blocks 2 and 3)
B,C,D (blocks 2, 3, and 4; note that if block 4 is admitted, blocks 2 and 3 must also be)
A, B, C, D (blocks 1, 2, 3, 4)

We will see, however, with a more detailed analysis that we can use blocks A, B, C and D, but eliminate additional tiles required for block 4, and have a ninth set, admitting just blocks 1, 2 and 3.

Block 1 substitution



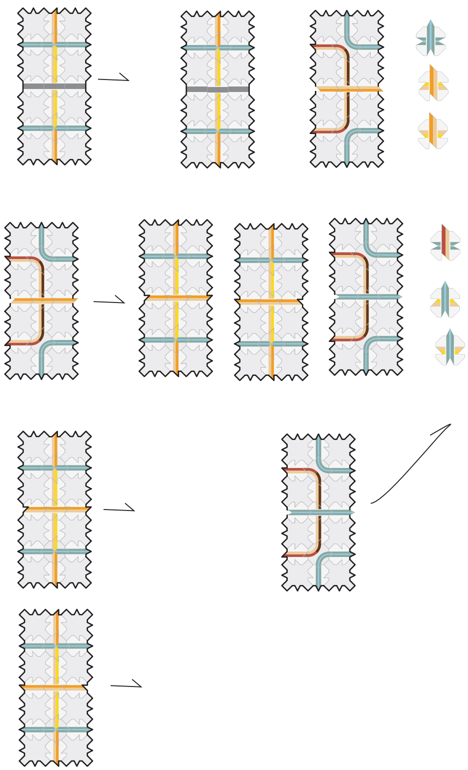
generates blocks



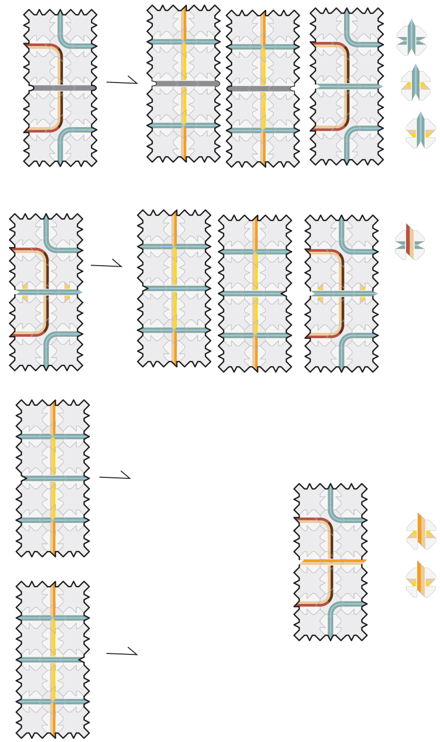
and additionally required tiles

Bl

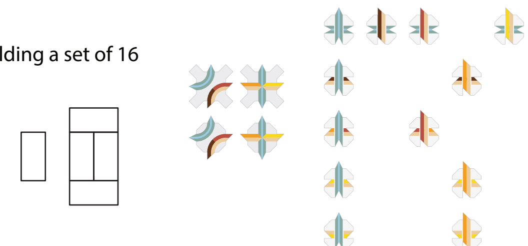
starting from block A requiring tiles

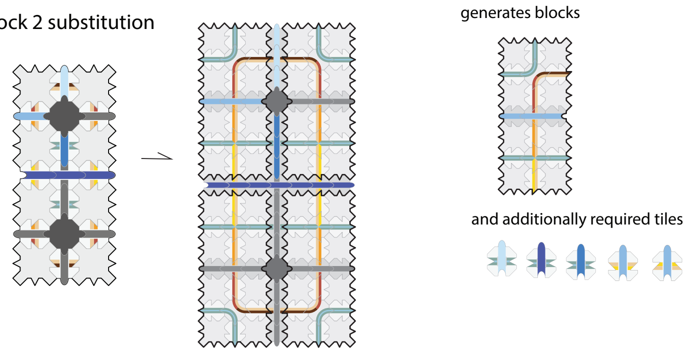


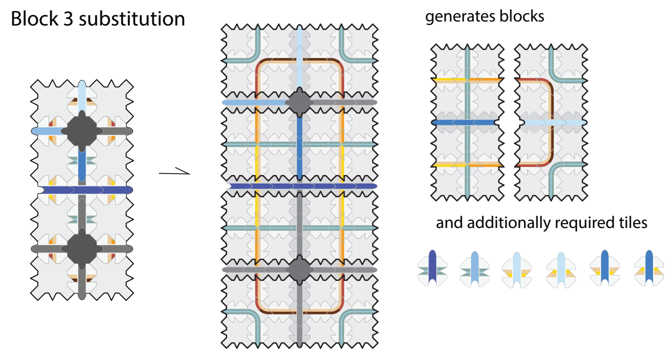
starting fromblock B, requiring tiles

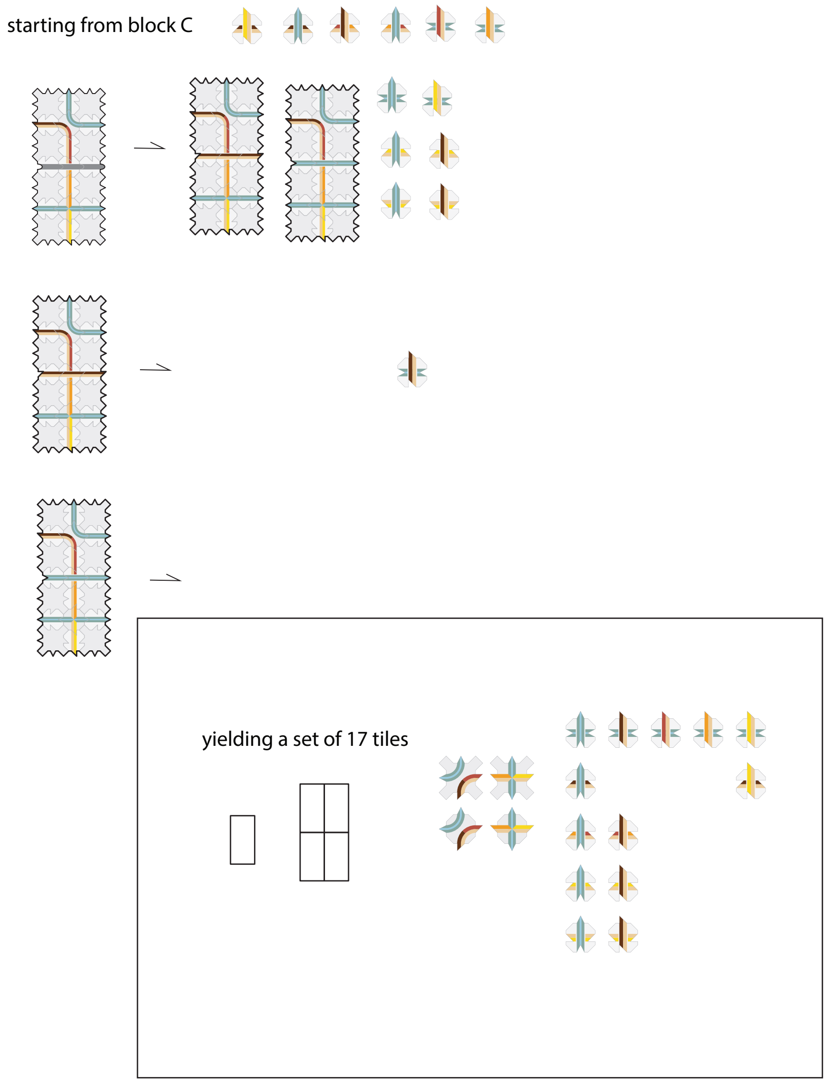


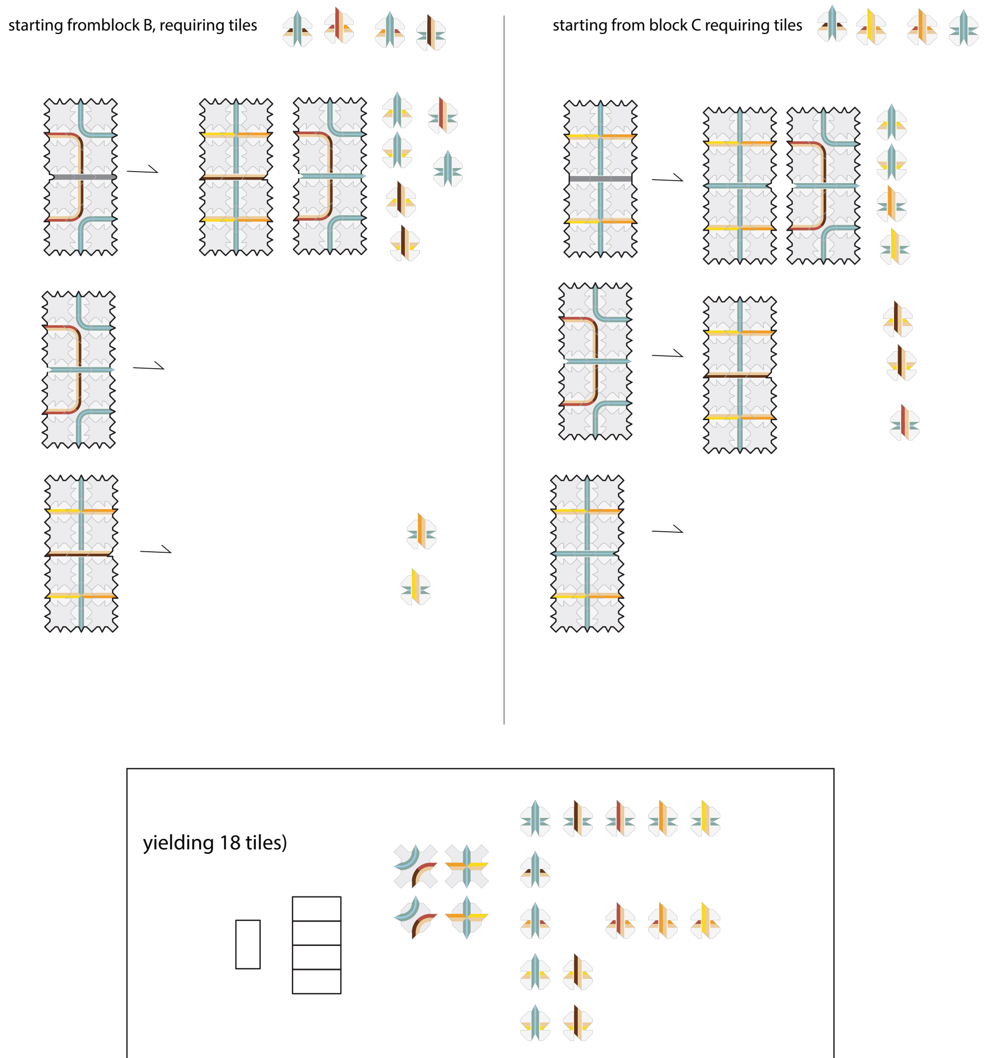
yielding a set of 16



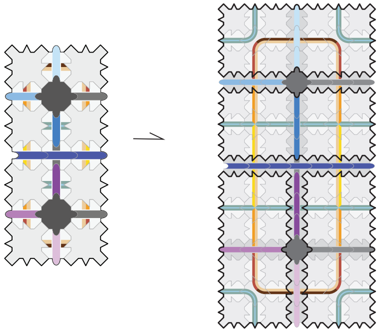

ock 2 substitution
generates blocks
and additionally required tiles


Block 3 substitution
generates blocks
and additionally required tiles

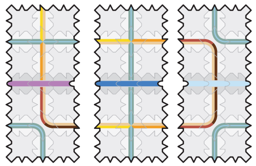

starting from block C
yielding a set of 17 tiles


starting fromblock B, requiring tiles
starting from block C requiring tiles
yielding 18 tiles)

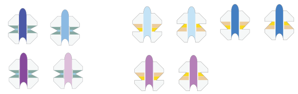
Block 4 substitution



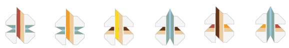
generates blocks

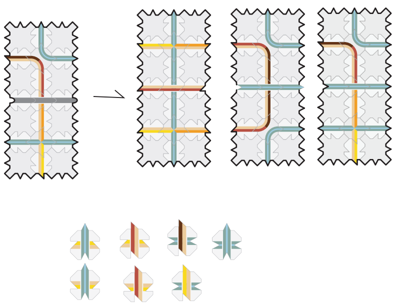


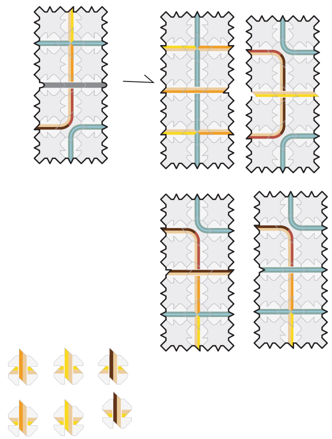
and additionally required tiles



starting from block C requiring tiles





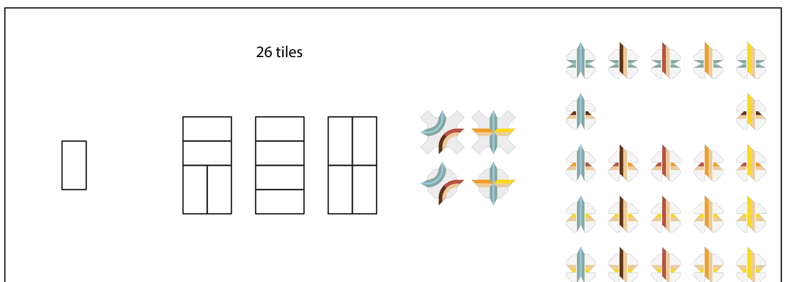


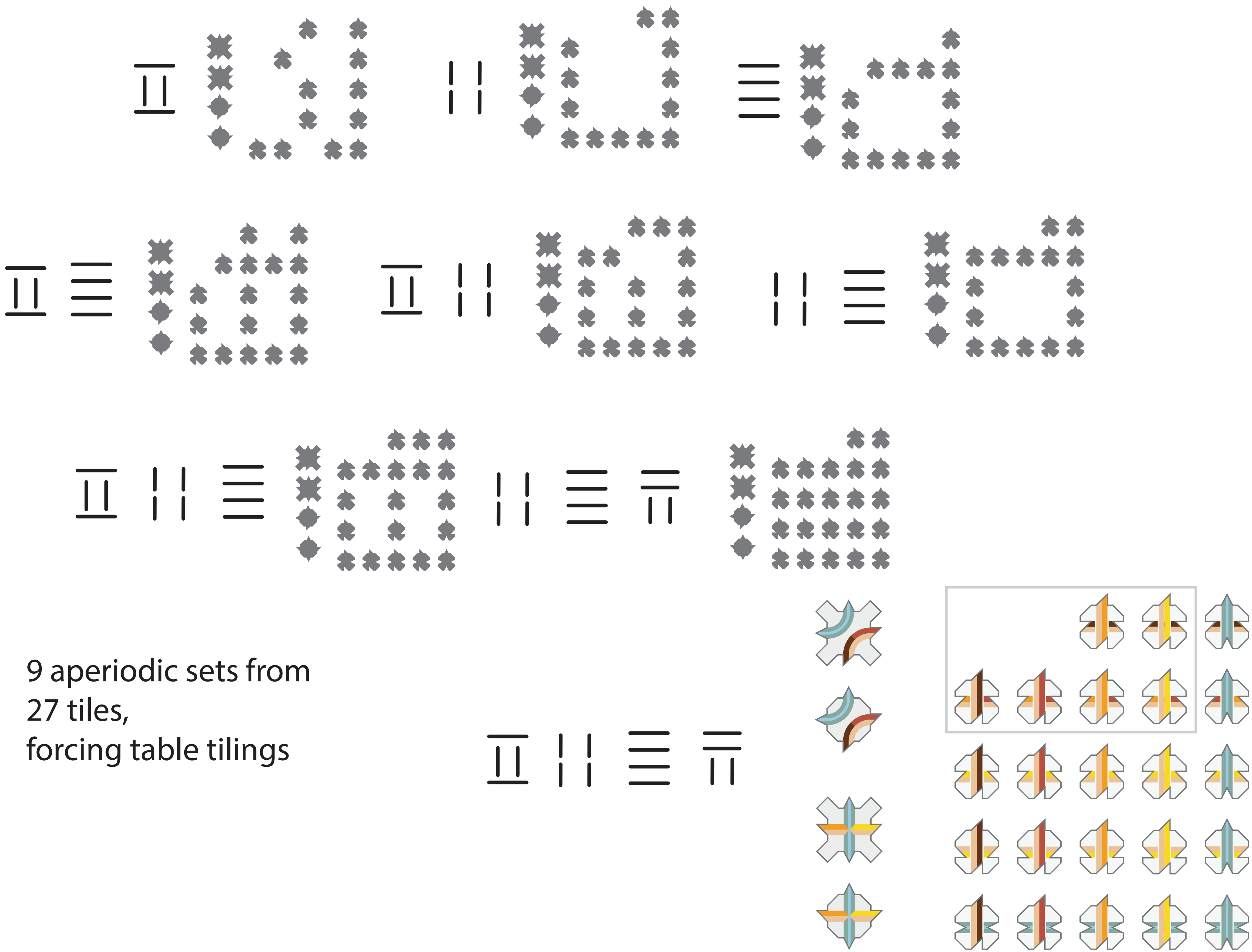
At this point, all blue dominated and yellow dominated tiles are required. Moreover, we must further use the tiles needed to assemble blocks C and D.

We will never need the brown dominating brown, or brown dominating red/orange.

Nor will we ever need orange dominating red/orange.

This gives a total of 26 tiles. Moreover, we must include the tiles that allow blocks 3 and 4 to be assembled, and we do not have control over the arrangements of tiles from one levelto the next.




9 aperiodic sets from
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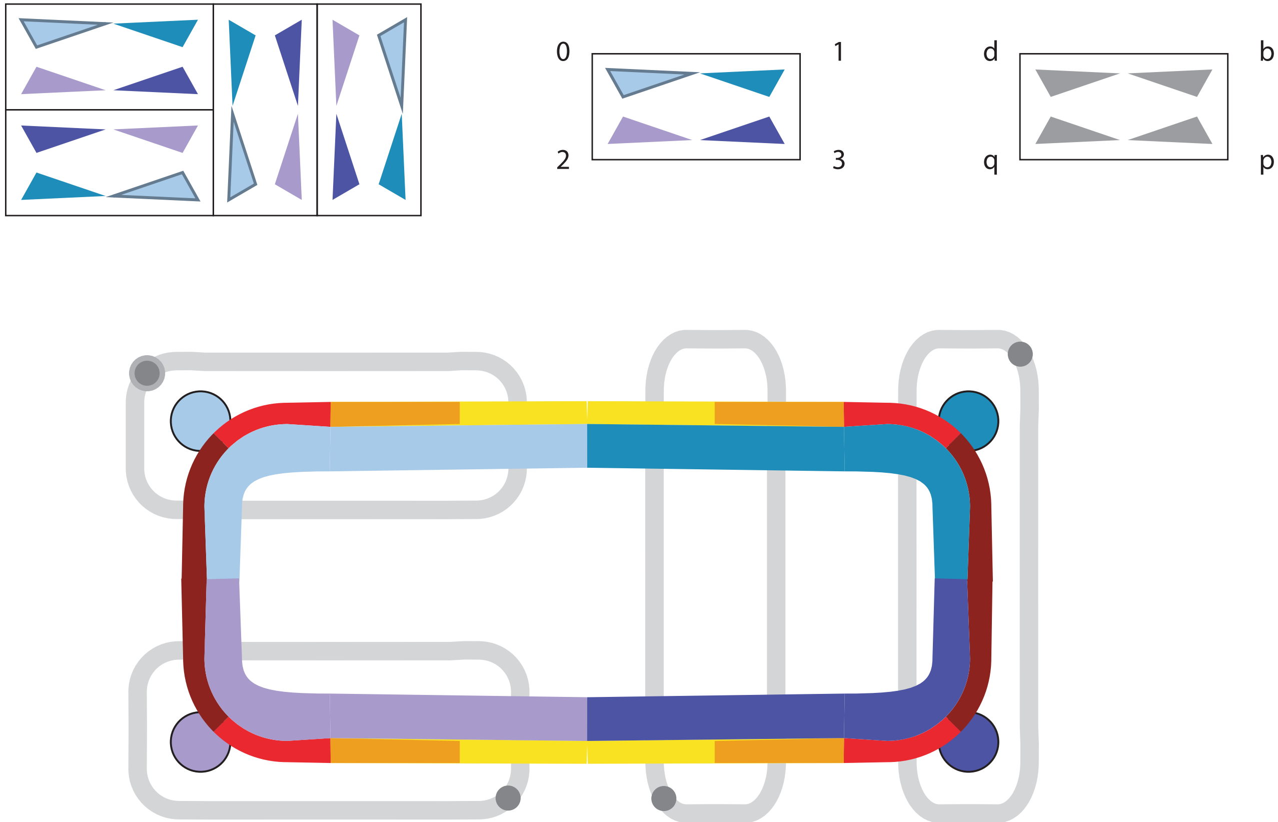
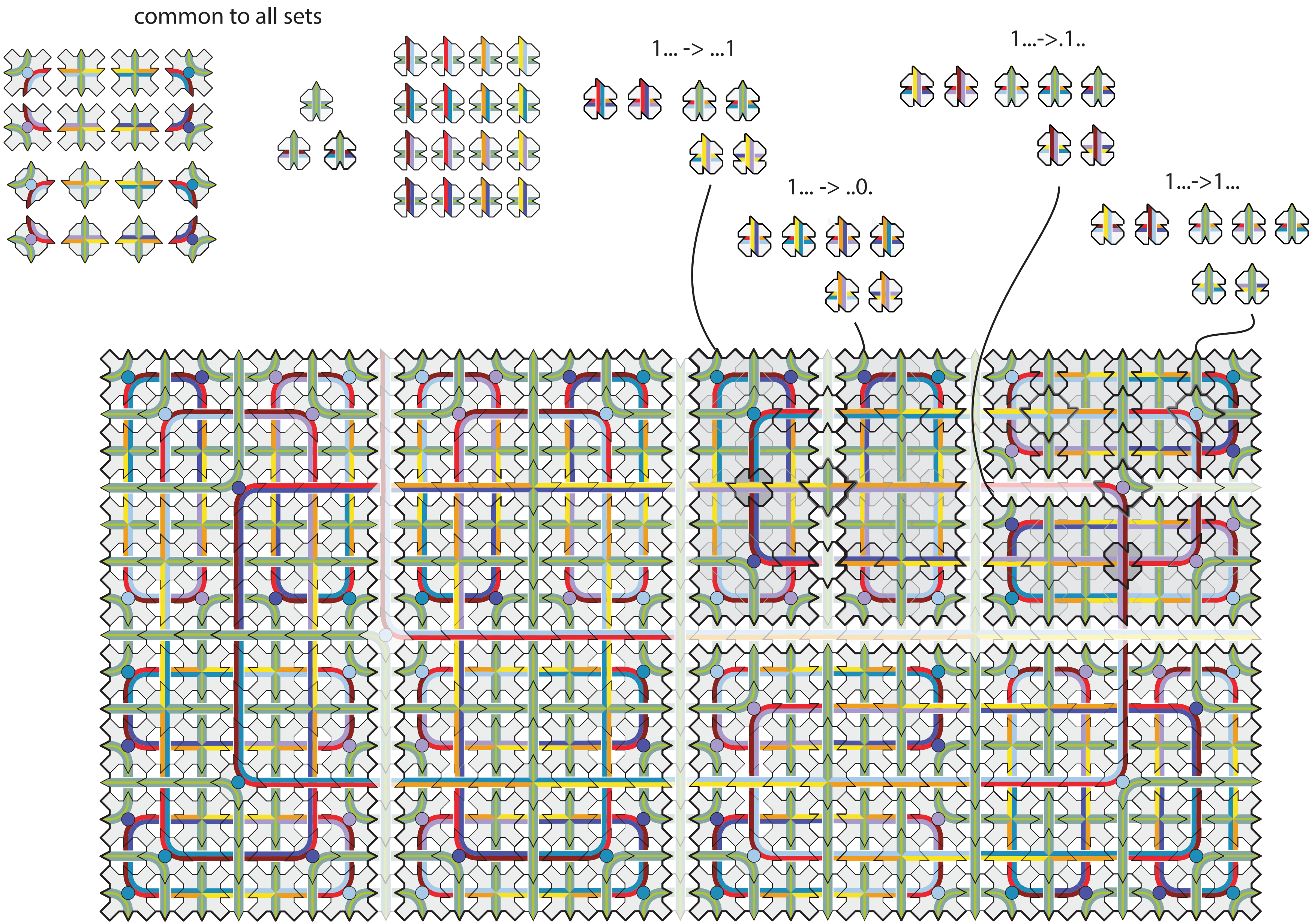
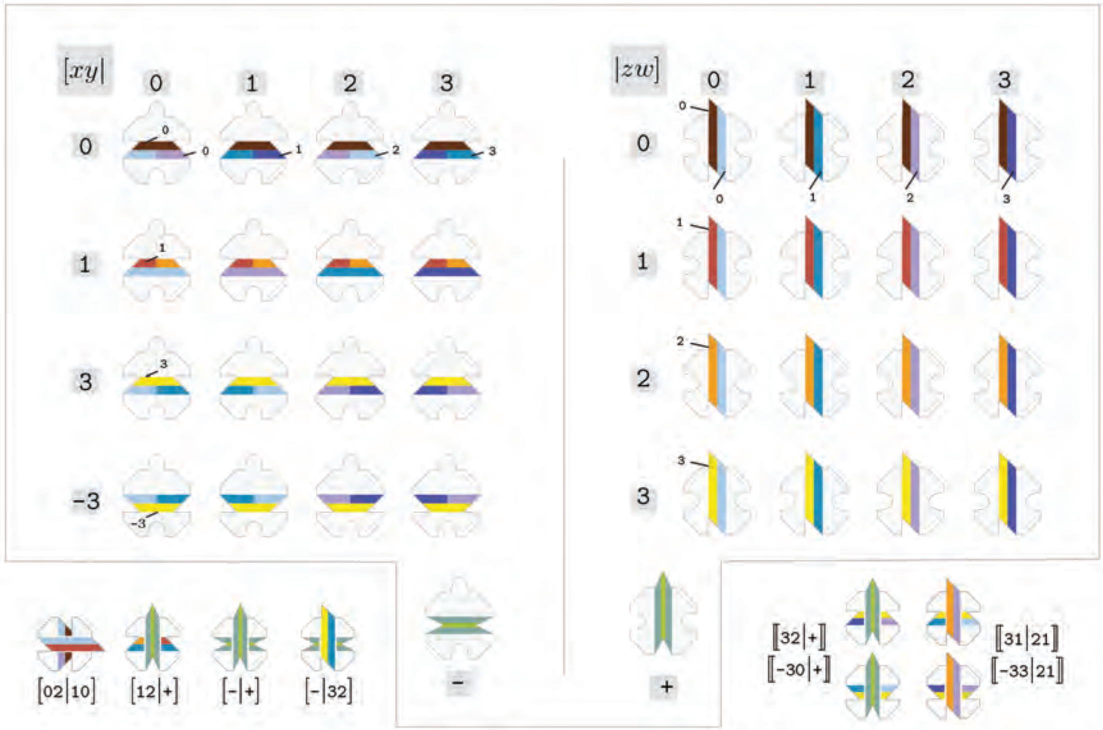
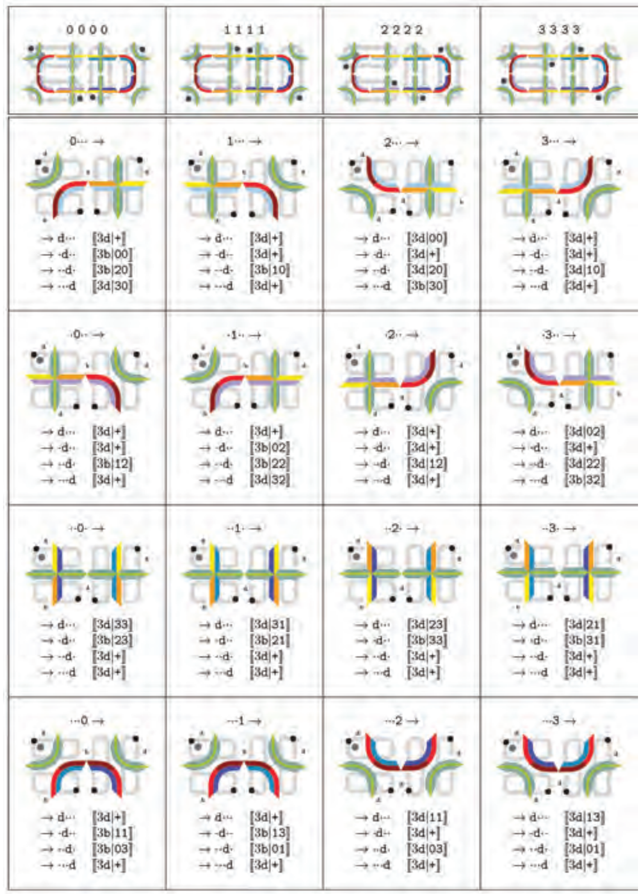

0
1
2
3
d
b
q
p

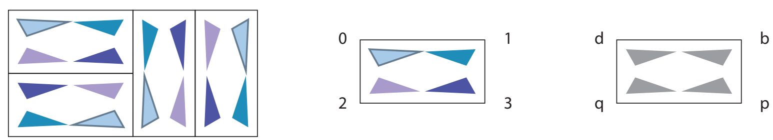
enforcing 1101

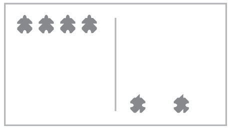
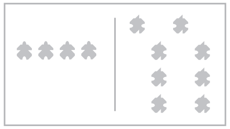


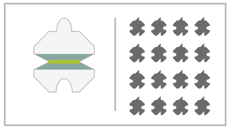
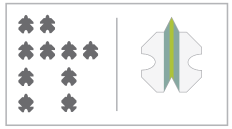
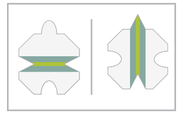


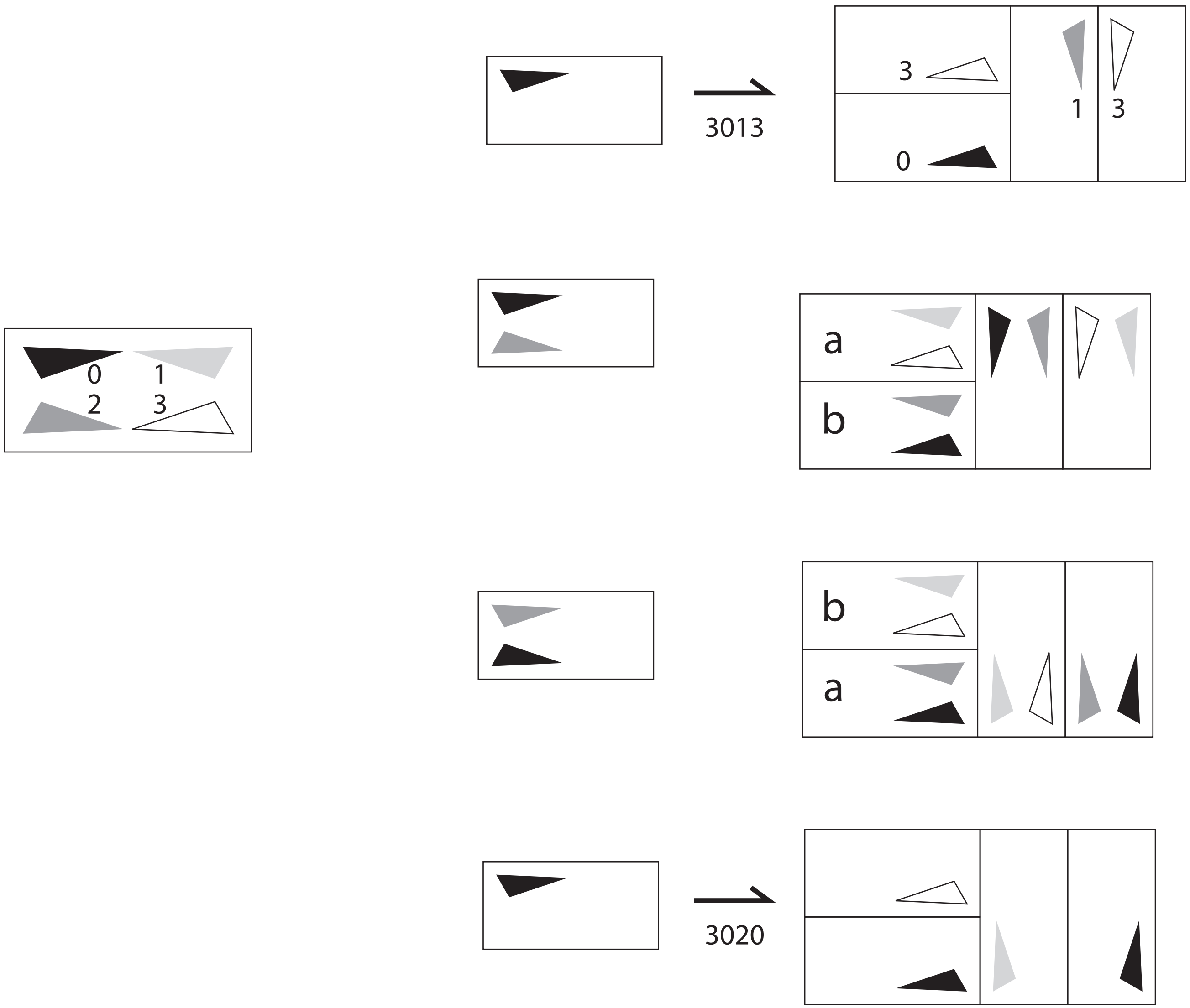


	(key tiles)	(not key tiles)
$\mathcal{T}_{\mathbf{d}\cdots} =$	{[1q|00], [0b|30],	[1d|+], [1b|+], [1p|+]}
$\mathcal{T}_{\cdot\mathbf{d}\cdot\cdot} =$	{[0d|32], [1p|02],	[1d|+], [1b|+], [1q|+]}
$\mathcal{T}_{\cdot\cdot\mathbf{d}\cdot} =$	{[1d|33], [1b|31], [1q|23], [1p|21]}	
$\mathcal{T}_{\cdots\mathbf{d}} =$	{[1q|11], [1p|13],	[1d|+], [1b|+]}



K  u 

 u  u  u 



abcd -> 3+bacd

Tiling Ratio