Height Functions for Hom shifts

undirected graph

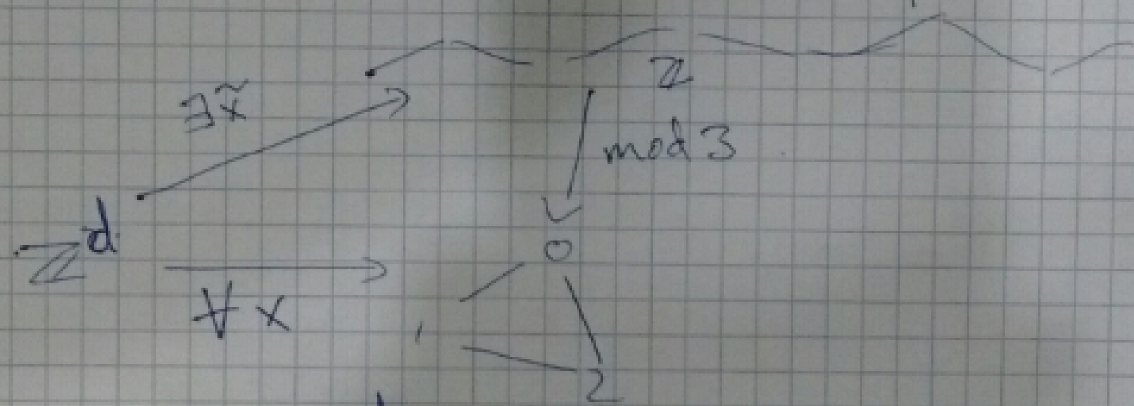
$$\mathrm{Hom}(\mathbb{Z}^d, H) = \{ x : \mathbb{Z}^2 \to H \mid \vec{i} \sim \vec{j} \Rightarrow x(\vec{i}) \sim_H x(\vec{j}) \}$$

$\mathbb{Z}^2$

K_n - complete graph with ~~respect to~~ n vertices $1, \dots, n$

$\mathrm{Hom}(\mathbb{Z}^d, K_n)$ - proper n-colourings of $\mathbb{Z}^2$.

$\mathrm{Hom}(\mathbb{Z}^d, 0 - 1)$ hard square model



$\forall x \in \mathrm{Hom}(\mathbb{Z}^d, \triangle_{0,1,2})$ $\exists \tilde{x} \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z})$

such that

$\tilde{x} \bmod 3 = x$

$\xrightarrow{\text{useful}}$ computing the entropy

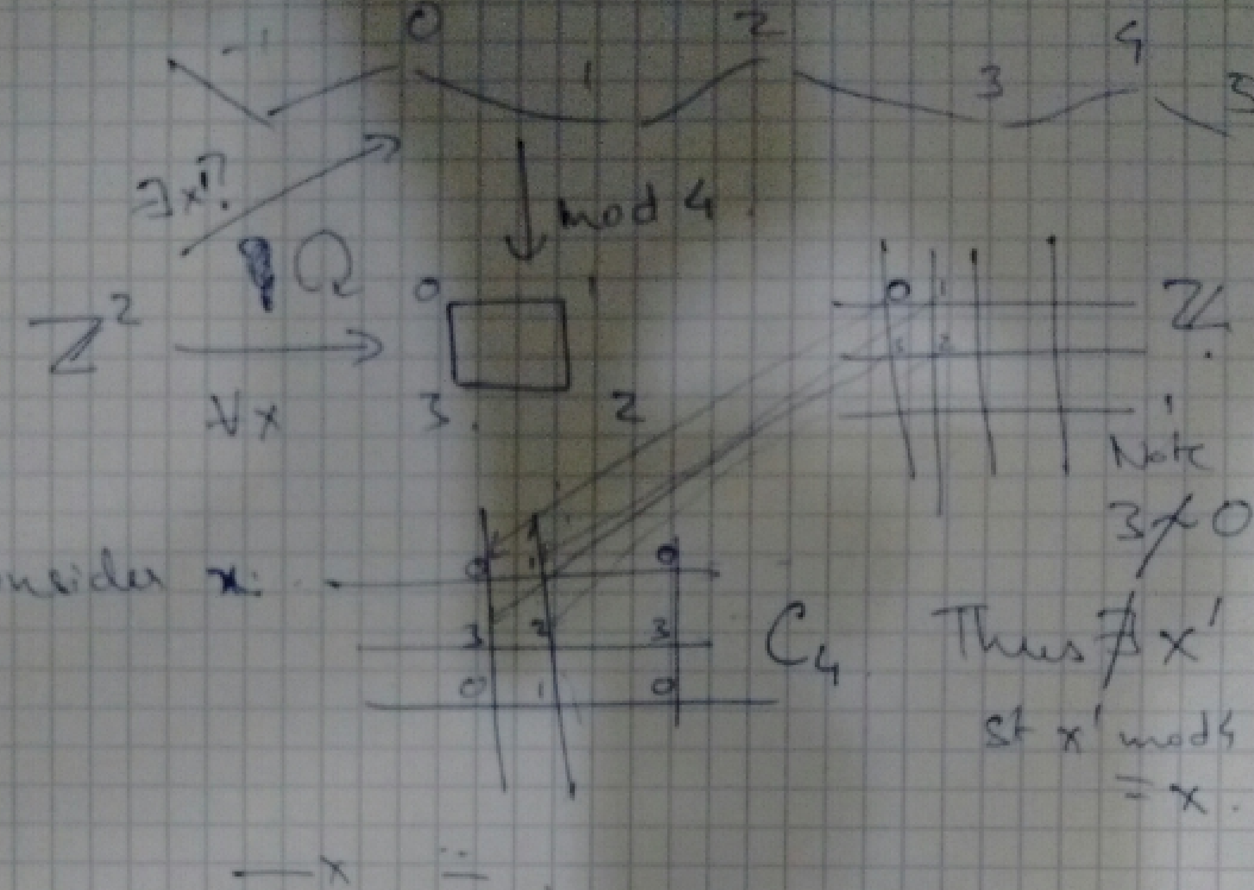
$\longrightarrow$ unique mme in 2 dimensions

$\longrightarrow$ multiple mmes in high enough dimension.

$\longrightarrow$ determines uniform mixing properties.

A shift space X is called SI if $\exists n$ st $\forall A, B \subset \mathbb{Z}^d$ $d(A,B) \geq n$, $x, y \in X$ $\exists z$ st $z|_A = x|_A$, $z|_B = y|_B$

$\longrightarrow$ $\mathbb{Z}$ infinite $\Rightarrow$ $\mathrm{Hom}(\mathbb{Z}^d, \triangle_{0,1,2})$ is not SI.



Consider x:

Thus $\nexists x'$ st $x' \bmod 4 = x$.

Ex:

Defn: $\pi: H' \to H$ covering map.

we say H lifts to H' if

the induced map $\pi: \mathrm{Hom}(\mathbb{Z}^d, H') \to \mathrm{Hom}(\mathbb{Z}^d, H)$

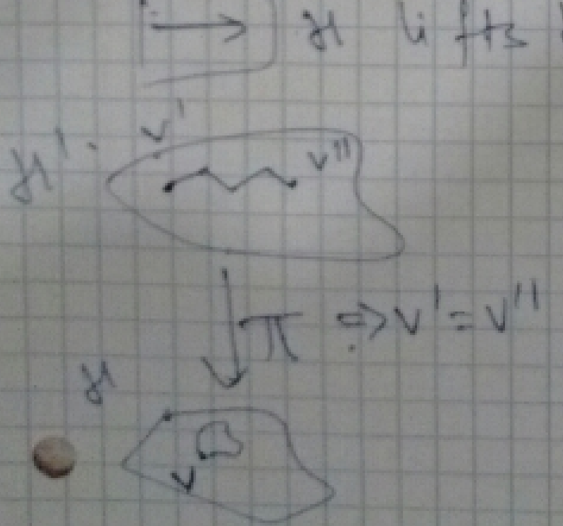
is surjective.

$\longrightarrow$ H lifts to H' iff for all walks in H'

v', v_1, v_2, v_3, v'' in H'

if $\pi(v') = \pi(v'')$

$\Rightarrow v' = v''$.



$\boxed{\Rightarrow}$ There is a maximal H'.

Call it H_{max}.

- How to obtain H_{max}?

$H_{max} \rightarrow$ (1) H

$\rightarrow$ (2) Construct M_H: stick 2-cells on all 4-cycles in H.

$\rightarrow$ (3) Consider universal cover of M_H.

$\rightsquigarrow M_{uni, H}$.

$\rightarrow$ (4) Remove 2-cells.

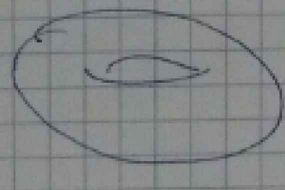
$\rightsquigarrow H_{max}$.

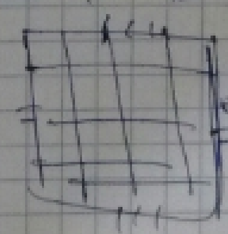
~~Tee~~

$\boxed{\Rightarrow}$ ~~Deg~~ $\mathrm{Deg}\,[H_{max} : H] = |\pi_1(M_H)|$

$\boxed{\Rightarrow}$ ~~$|H_{max}| = \infty \Rightarrow H_{max}(\mathbb{Z}^2, H)$ is~~

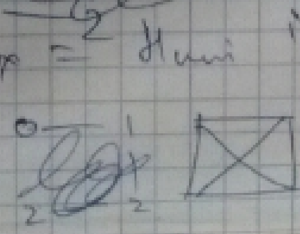
Ex.:



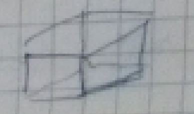


$= H \rightsquigarrow H_{max} = \mathbb{Z}^2$

$\boxed{\Rightarrow}$ $H'_{max} = H_{uni}$ iff H has no four-cycles



$\xrightarrow{H_{max}}$



~~Given a finitely present~~

$\Rightarrow$ Given any finitely presentable groups G.

$\exists\ H$ st $\pi_1(M_H) = G$ & has

~~Deg([~~

$\mathrm{Deg}([H_{max} : H]) = |G| \to$

$\boxed{\to}$ It is undecidable whether

$\to H_{max} = H$.

$\to |H_{max}| = \infty$

Cor: It is undecidable whether ~~H_{max}~~

$H_m(\mathbb{Z}^n, H)$ = is S^1.

$\to |H_{max} : H| \to$ gives

a ~~quot~~ ~~general~~

topological

constraint

$\leadsto$ there is a more

quantitative

obstruction

which is

harder to

understand