Geometric and dynamical features of quasicrystals

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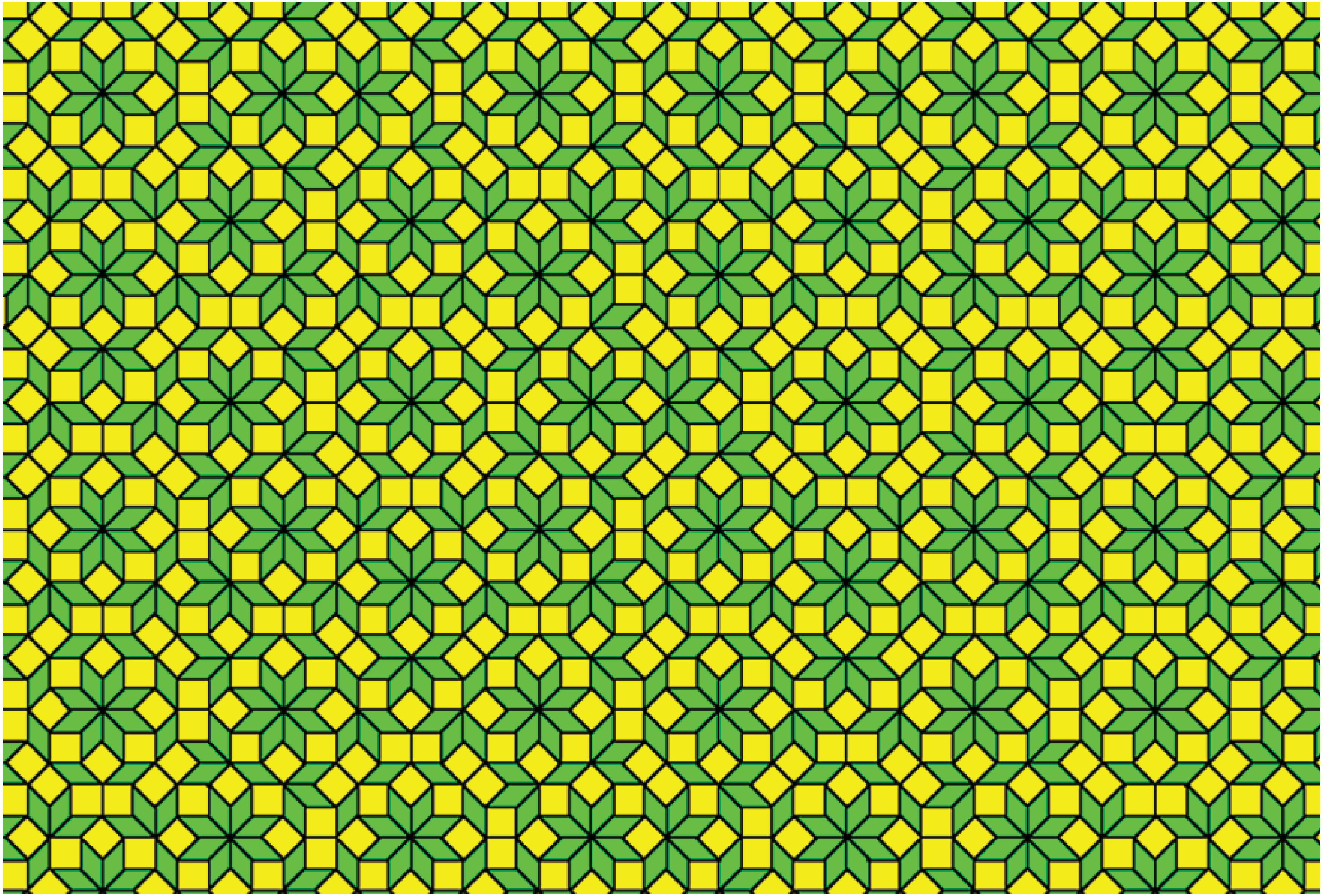

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0. Introduction

Point of view of mathematics: Aperiodic order.

Point of view of physics: Quasicrystals.

1. What is aperiodic order?

2. Why study aperiodic order?

3. How to study aperiodic order?

0.1. What is aperiodic order?

Aperiodic order is a specific form of long range order beyond lattice structures. There is no axiomatic framework (yet). Typical features include:

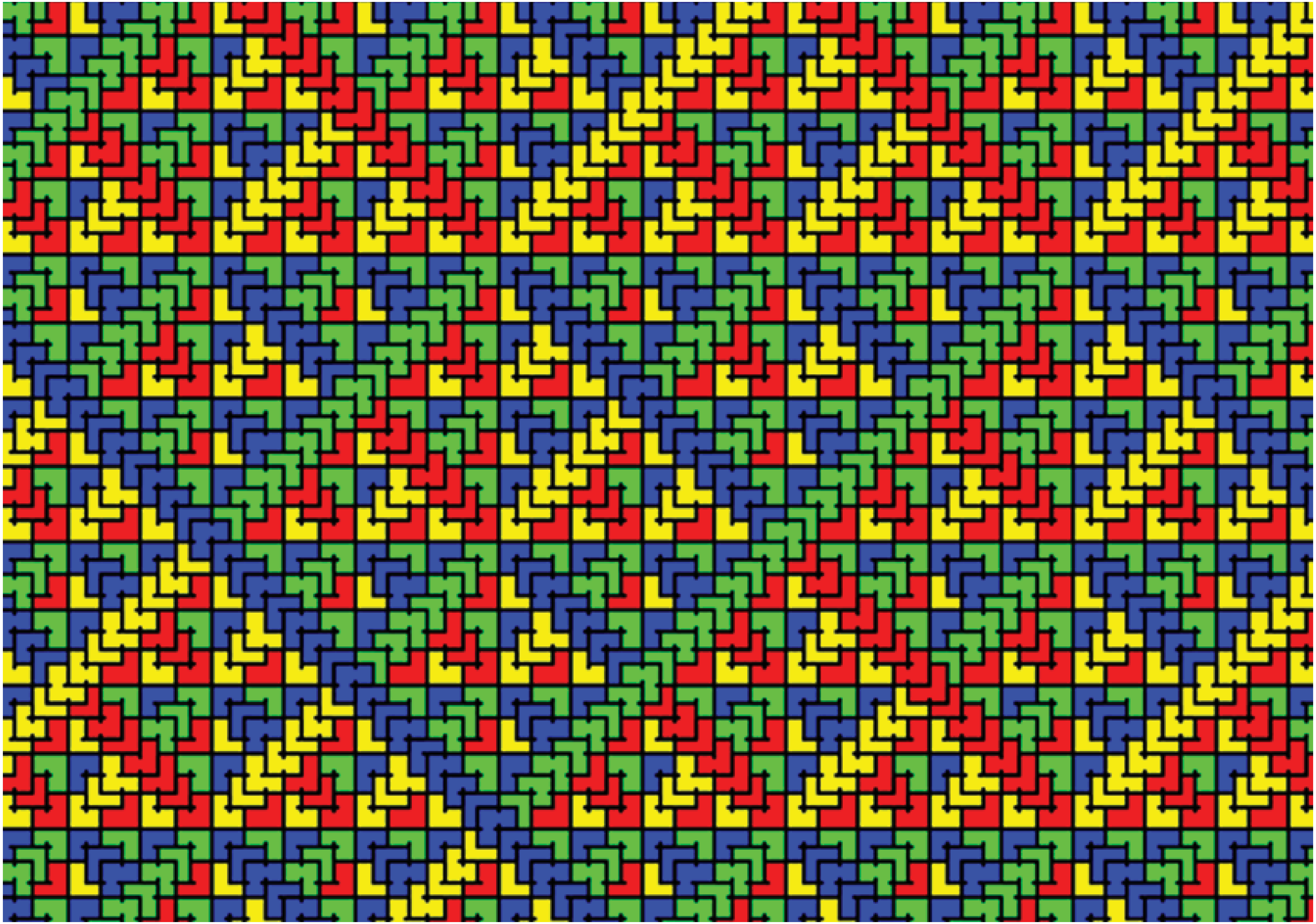
- Point diffraction ($-- >$ experiment).

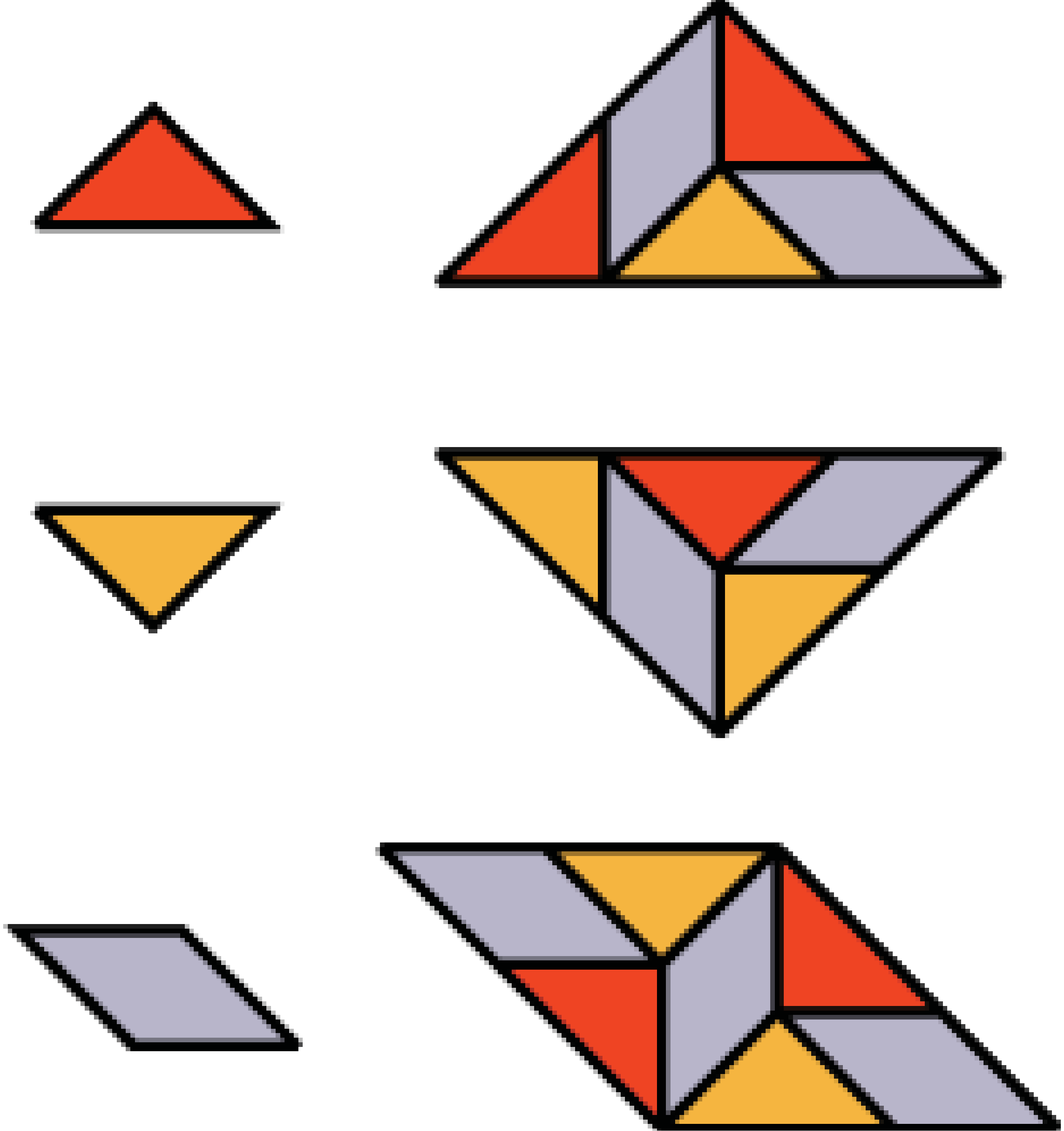
- Low complexity (zero entropy).

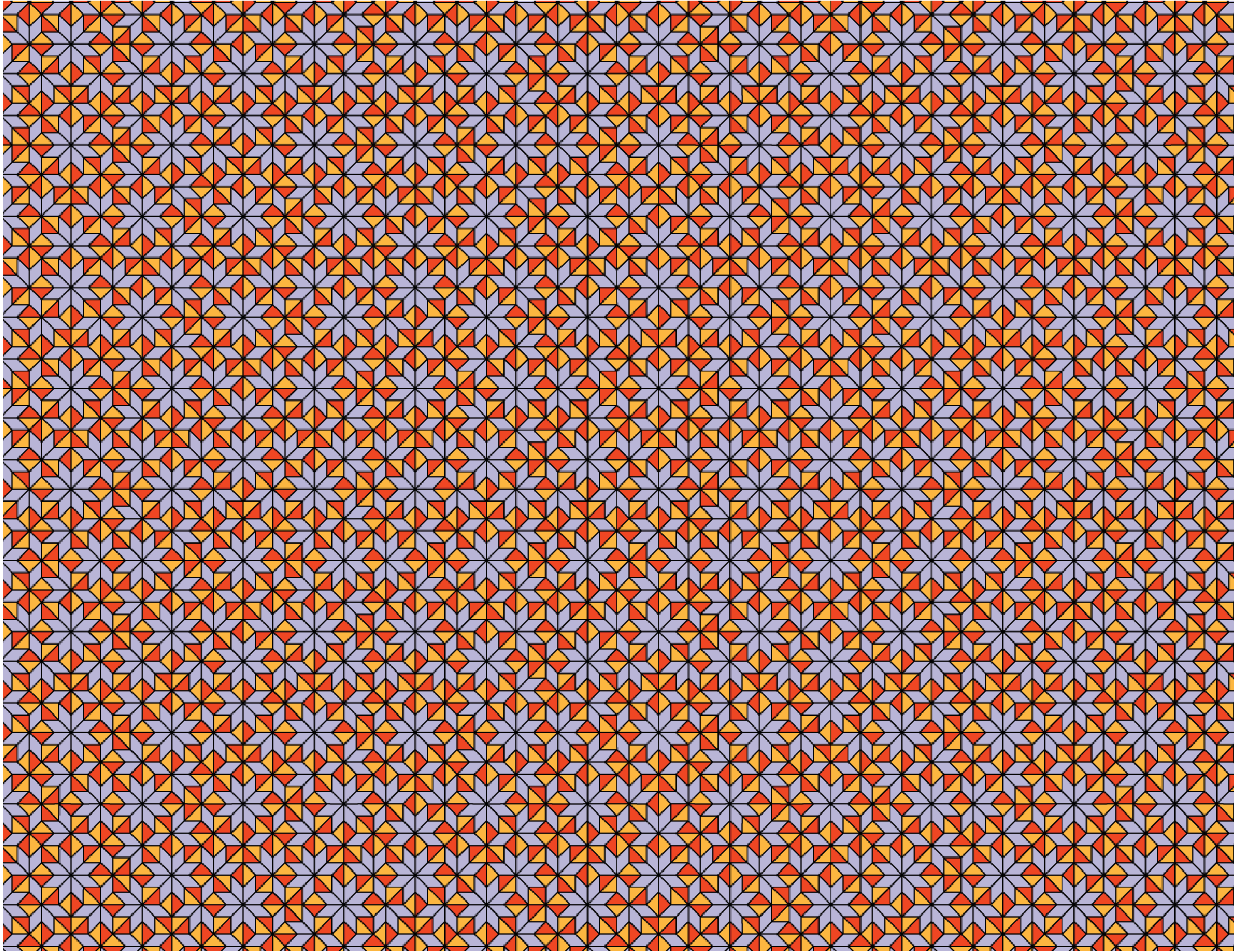
- Repetitivity.

- Deterministic rules.

Note: Aperiodic order is not about small parameters.







The Fibonacci model

$$S:\{0,1\} \longrightarrow \{0,1\}^*,\;\; S(0)=01,\; S(1)=0.$$

$$\begin{aligned}
S^0(0) &= & & = 0\\
S^1(0) &= S(0) &&= S(0) &&= 01\\
S^2(0) &= S(S(0)) &&= S(01) &&= 010\\
S^3(0) &= S(S^2(0)) &&= S(010) &&= 01001\\
S^4(0) &= S(S^3(0)) &&= S(01001) &&= 01001010\\
S^5(0) &= S(S^4(0)) &&= S(01001010) &&= 0100101001001
\end{aligned}$$

$$S^{n+1}(0) = S^n(S(0)) = S^n(0)\,S^{n-1}(0).$$

Infinite word: $\omega = \lim_{n\to\infty} S^n(0).$

0.2. Why study aperiodic order?

- Study of disorder (and related phenomena) is one of the most important issues in mathematics and physics today!

- Aperiodic order is a specific form of intermediate disorder of substantial conceptual interest. (Morse / Hedlund '42, ..., Lagarias '99, Lagarias/Pleasants '03, ...)

- Fourier expansion of sets (Y. Meyer '72, ...)

- Quasicrystals (Shechtman/Blech/Gratias/Cahn '82 (published '84), Nobel prize in Chemistry for Shechtman in 2011)

  - Sharp diffraction peaks $< - - - >$ long range order.

  - 5-fold symmetry $< - - - >$ no lattice.

0.3. How to study aperiodic order?

- FLC Delone sets / finite set of tiles : complexity and geometric features.

- Translation bounded measures: generalized almost periodicity.

Important concept **hull** : topological space and dynamical system. Gives powerful methods to study aperiodic order.

In one-dimensional case: Sequences and subshifts over a finite alphabet.

I. FLC Delone sets and discrete geometry.

1. Basic concepts for point sets in $\mathbb{R}^N$.

2. Lattices and crystallographic sets.

3. FLC-Delone sets and Lagarias' theorem.

4. Repetitivity.

5. Aperiodicity and repulsion.

6. Linear repetitivity.

7. Patch counting, dense repetitivity and vanishing entropy.

8. Uniform patch frequency.

9. Meyer-Moody-theory and a result of Lagarias.

I.1. Basic concepts for point sets in $\mathbb{R}^N$.

$\varLambda \subset \mathbb{R}^N$ is called

- *uniformly discrete* if there exists a $\sigma > 0$ with

  $$U_\sigma(x) \cap U_\sigma(y) = \emptyset$$

  for all $x, y \in \varLambda$ with $x \neq y$.

- *relatively dense* if there exists a $\varrho > 0$ with

  $$B_\varrho(x) \cap \varLambda \neq \emptyset$$

  for all $x \in \mathbb{R}^N$.

- *Delone set* if it is both uniformly discrete and relatively dense.

Point sets versus tiling. We talk about point sets but show pictures of tilings. Abstract background:

Tiling $---$ > *point set*: Take in each tile the center of mass. This gives a point set with similar properties.

Point set $---$ > *tiling*: To each $x \in \Lambda$ consider its *Voronoi cell*

$$V(x) := \{p \in \mathbb{R}^N : \|p-x\| \leq \|p-y\| \text{ all } y \in \Lambda\}.$$

These Voronoi cells give a tiling of $\mathbb{R}^N$ consisting of convex polytopes.

These two procedures are 'morally' inverse to each other.

I.2. Lattices and crystallographic sets.

Lattices are the paradigm of order in our context. Aperiodic order will be about structures with weak lattice type properties.

Definition. A *lattice* in $\mathbb{R}^N$ is a discrete subgroup with compact quotient.

Alternatively, $\Gamma \subset \mathbb{R}^N$ lattice means:

- Γ uniformly discrete.
- Γ relatively dense.
- Γ subgroup of $\mathbb{R}^N$.

$\Gamma \subset \mathbb{R}^N$ is a lattice if and only if there exists a basis $a_1, \ldots, a_N$ of $\mathbb{R}^N$ with

$$\Gamma = \mathsf{Lin}_{\mathbb{Z}}\{a_j : j = 1, \ldots, N\} = (a_1, \ldots, a_N)\mathbb{Z}^N.$$

If Γ is a lattice then so is

$$\Gamma^* := \{k \in \mathbb{R}^N : e^{2\pi kl} = 1 \text{ for all } k \in \Gamma\}.$$

Then, Γ^* is called the *dual lattice*.

$\Lambda \subset \mathbb{R}^N$ is called *crystallographic* if it is uniformly discrete and invariant under a lattice Γ.

Thus, Λ crystallographic if and only if

$$\Lambda = F + \Gamma$$

with a lattice Γ and F finite.

I.3. FLC-Delone sets and Lagarias' theorem.

A piece of philosophy.

$\Gamma \subset \mathbb{R}^N$ is a subgroup of $\mathbb{R}^N$ if and only if

$$\Gamma - \Gamma \subset \Gamma.$$

Consider how far general Delone set $\Lambda \subset \mathbb{R}^N$ is from being a lattice by investigating $\Lambda - \Lambda$.

In order for this to make sense we will need at the very least a certain finiteness property of $\Lambda - \Lambda$.

Lemma (Lagarias '99, Schlottmann '00). *For $\varLambda \subset \mathbb{R}^N$ uniformly discrete the following assertions are equivalent:*

(i) $\sharp(\varLambda - \varLambda) \cap B_R(0) < \infty$ *for every* $R > 0$.

(ii) $\varLambda - \varLambda$ *is discrete and closed.*

(iii) $\sharp\{(\varLambda - x) \cap B_R(0) : x \in \varLambda\} < \infty$ *for all* $R > 0$.

A Delone set $\varLambda \subset \mathbb{R}^N$ is said to be of *finite local complexity* (FLC) if it satisfies one of the equivalent conditions of the previous lemma. Such a $\varLambda$ is also called an *FLC-Delone set*.

Remark. Typically sets generated by stochastic processes are not Delone sets of finite local complexity.

The concept of patch.

$\Lambda \subset \mathbb{R}^N$ Delone set: A *patch* of size R of Λ is a set of the form

$$P(x, R) := (\Lambda - x) \cap B_R(x)$$

for $x \in \Lambda$.

Clearly, Λ Delone has finite local complexity if and only if for any $R > 0$ there exist only finitely many patches of size R.

The one-dimensional case.

Sets of finite local complexity in $\mathbb{R}$ can be generated as follows:

- Take a finite set of intervals in $\mathbb{R}$.

- Cover $\mathbb{R}$ by translates of these intervals.

- Take the boundary points of the intervals.

Then these boundary points will be an FLC-Delone set.

The converse is true as well:

Let $\varLambda$ be an FLC-Delone set in $\mathbb{R}$ with $\varLambda \cap B_\varrho(p) \neq \emptyset$ for all $p \in \mathbb{R}$.

By the relative denseness condition, distances between 'neighbours' will not exceed 2ϱ.

By FLC, there will then only be finitely many elements in

$$L := \{y - x : y \text{ neighbour of } x \text{ with } x < y\}.$$

Now, clearly, $\varLambda$ arises from tiling $\mathbb{R}$ with tiles of the form $[0, l)$ with $l \in L$.

A higher dimensional analogue is valid as well:

Theorem (Lagarias '99). *Let $\Lambda \subset \mathbb{R}^N$ be a Delone set and $\varrho > 0$ with*

$$\Lambda \cap B_\varrho(p) \neq \emptyset$$

for any $p \in \mathbb{R}^N$. Then, the following assertions are equivalent:

(i) Λ is of finite local complexity.

(ii) $\sharp\{(\Lambda - x) \cap B_{2\varrho}(0) : x \in \Lambda\} < \infty$

Consequence. We may think of Delone sets of finite local complexity as geometric analogues to sequences over a finite alphabet (with the alphabet corresponding to patches of size 2ϱ).

Finite local complexity is not in itself a sign of order. (Compare coin tossing experiment and Bernoulli subshift.)

However, it allows one to define and study notions of order.

This is done next.

I. 4-8: Notions of order for FLC-Delone sets.

We will now meet (and study) various concepts of order and aperiodicity in FLC-Delone sets:

- Repetitivity.

- Aperiodicity and repulsion.

- Linear repetitivity.

- Patch counting, dense repetitivity and vanishing entropy.

- Uniform patch frequency.

I.4. Repetitivity

Given $\Lambda \subset \mathbb{R}^N$ FLC-Delone set, P patch of size R of Λ i.e. P is of form

$$P := (\Lambda - x) \cap B_R(x)$$

for some $x \in \Lambda$.

The *locator set of* P also known as *derived Delone set with respect to* P is defined as

$$\Lambda_P := \{y \in \Lambda : (\Lambda - y) \cap B_R(y) = P\}.$$

Definition (Repetitivity)**.** *An FLC-Delone set is called repetitive if the locator set of any of its patches is relatively dense (and, hence, a Delone set as well).*

Remark. Any lattice and any crystallographic set is repetitive.

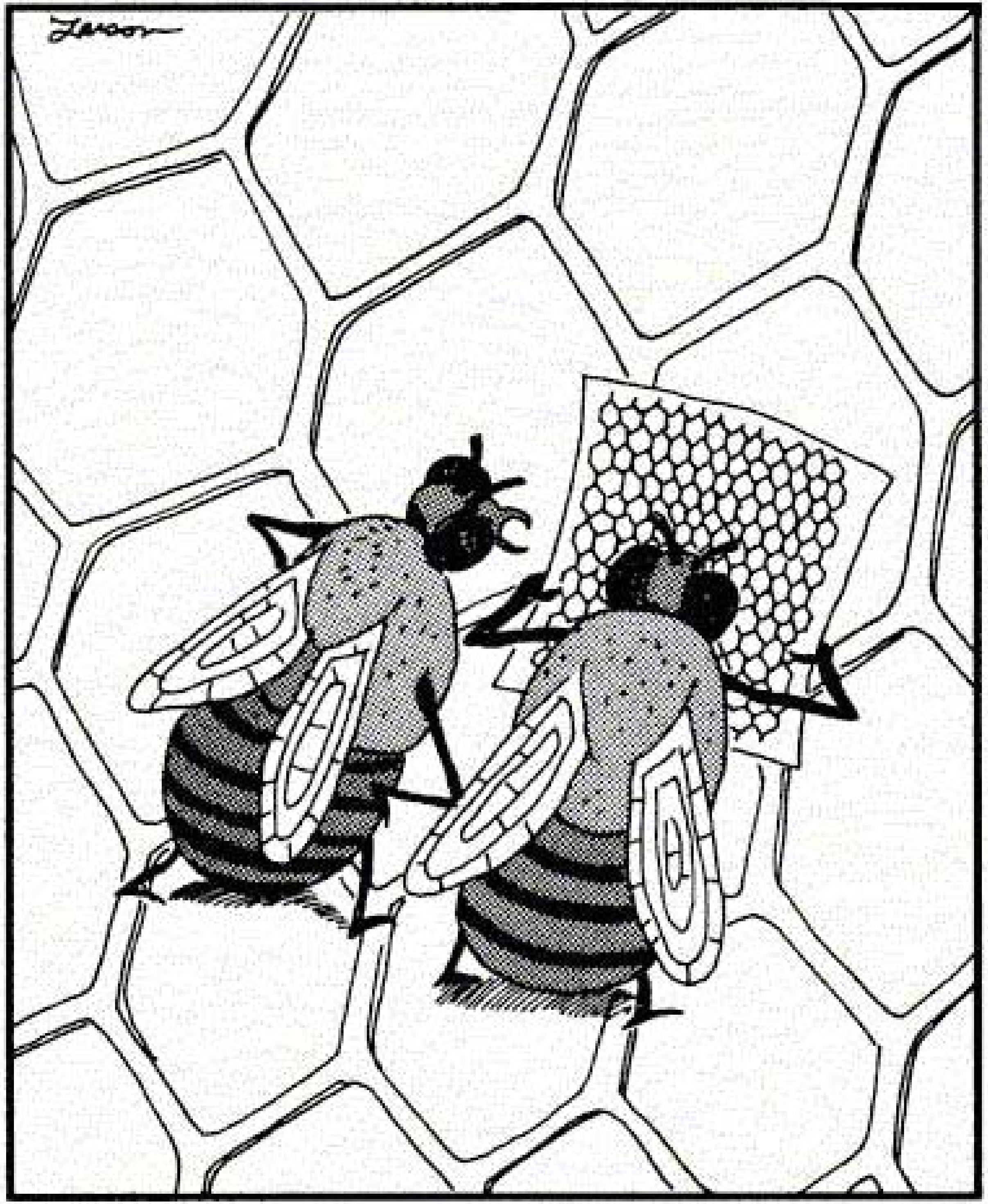
FLC-Delone set is repetitive if and only if for any patch P there exists a radius $\varrho(P) > 0$ such that

$$B_{\varrho(P)}(p) \cap \Lambda_P \neq \emptyset$$

for any $p \in \mathbb{R}^N$, i.e. such that any ball of radius $\varrho(P)$ contains the center of a copy of P.

Remark. There is no relation between the size R of the patch P and $\varrho(P)$.

Here is what repetitivity means:



"Face it, Fred—you're lost!"

The repetitivity function

Λ FLC-Delone set. Define

$$\mathrm{Rep}_\Lambda : [1,\infty) \longrightarrow [0,\infty), \mathrm{Rep}_\Lambda(R) := \{\max \varrho(P) : P \text{ patch of size } R\}.$$

Remark. We need to impose a lower bound on the possible values of R in order to deal with the patch consisting of one point only.

Note. If Λ is crystallographic then Rep_Λ is bounded. (Converse holds as well, as we will see shortly.)

I.5. Aperiodicity and repulsion

An FLC-Delone set *admits the period* $p \in \mathbb{R}^N$ if $\Lambda + p = \Lambda$.

Lemma (Repulsion property). *Let Λ be repetitive FLC-Delone set. If Λ does not admit a non-trivial period, then $\varrho(P_n) \to \infty$ for any sequence of patches P_n with sizes $R_n \to \infty$.*

Remark. Without the assumption of repetitivity, the statement does not hold. (Consider e.g. $2\mathbb{Z} \cup \{1\} \subset \mathbb{R}$....)

Corollary. *Let Λ be a repetitive FLC-Delone set. If Λ does not admit a non-trivial period, then $\mathrm{Rep}_\Lambda(R) \to \infty$ for $R \to \infty$.*

Remark. We may call a repetitive FLC-Delone set *aperiodic* if it does not admit a non-trivial period.

A lower bound for the repetitivity function

Theorem (Lagarias / Pleasants '03, '02.)**.** *Let Λ be a repetitive FLC-Delone set. If Λ satisfies*

$$\mathit{Rep}_{\Lambda}(R) \leq \frac{R}{3}$$

for some $R > 0$, then Λ is crystallographic.

Remark - History. Corresponding result for subshifts due to Morse / Hedlund '38 (without the constant 3).

Aperiodic order and the gap

There is a gap between

- the ordered world having bounded Rep and
- the non-ordered world having at least linear growth of Rep.

We have met this gap already in the symbolic case in the Morse / Hedlund theorem and we are going to meet it again.

I.6. Linear repetitivity

The previous result suggests to single out the following class of models.

Definition (Linear repetitiv). *Let Λ be a Delone set of finite local complexity. If there exists a $C > 0$ with*

$$\mathit{Rep}_\Lambda(R) \leq CR$$

for all $R \geq 1$, then Λ is called linearly repetitive.

Thus, Λ is linearly repetitive if there exists $C \geq 0$ such that for any patch P with radius $R \geq 0$ we have

$$B_{CR}(y) \cap \Lambda_P \neq \emptyset$$

for any $y \in \mathbb{R}^N$.

Remark. Linearly repetitive sets were brought forward by Lagarias / Pleasants '99 to model perfectly ordered quascicrystals, see below for further history.

Partial history

In the symbolic case:

- studied by Boshernitzan in the 90ies (unpublished);

- introduced under the name *linear recurrence* by Durand / Host / Skau '99;

- thorough study and characterization by Durand '00;

- appears under the name of *window property* in Damanik / Zare '00 in the context of primitive substitutions.

Partial history - continued.

In the tiling / Delone situation:

- appears in Solomyak '98 under the name *uniform repetitivity* in the context of primitive substitution.

- featured by Lagarias / Pleasants '03 (preprint from '99) under the name *linear repetitivity*.

I.7. Patch counting, dense repetitivity and vanishing entropy

Patch counting function gives notion capturing order features:

$$p_{\Lambda}(R) := \sharp\{P : \; P \text{ is patch of size } R \text{ in } \Lambda\}.$$

General idea: Slow growth of p_{Λ} means some form of order.

Remark. If Λ is crystallographic then p_{Λ} is bounded. Converse holds as well; see next theorem.

Theorem (Lagarias / Pleasants '03). *Let Λ be an FLC-Delone set with parameter ϱ of relative denseness. If there exists $R > 0$ with $p_\Lambda(R) < \frac{R}{\varrho}$ then Λ is crystallographic.*

Remark. (a) Analogue to a result of Morse / Hedlund '38 in the symbolic case. (Minimal word complexity is $n + 1$ in the aperiodic case.)

(b) It was conjectured by Lagarias / Pleasants '03 that there exists $c = c(\sigma, \varrho, N)$ such that

$$\frac{p_\Lambda(R)}{R^{N-k+1}} < c$$

for all sufficiently large R implies that Λ has at least k linearly independent periods. The theorem proves the case $k = N$. The general case was *disproved* by Cassaigne '06.

(c) Compare discussion Nivats conjecture and recent results of Kra / Cyr.

Connections between $\mathbf{Rep}_\Lambda$ and p_Λ.

Theorem (Lagarias / Pleasants '03). *Let Λ be a repetitive FLC-Delone set. Then,*

$$\mathit{Rep}_\Lambda(R) \geq \sigma \cdot ((p_\Lambda(R))^{1/n} - 1).$$

Remark. As shown by Lagarias / Pleasants '03, there is no upper bound (even for FLC-Delone sets Λ with polynomial growth of p_Λ coming form cut and project schemes).

Densely repetitive sets. The previous results suggests to single out the following class of sets:

Definition. *A repetitive FLC-Delone set Λ is called densely repetitive if there exists a $C > 0$ with*

$$\mathit{Rep}_\Lambda(R) \leq C \cdot (p_\Lambda(R))^{1/n}$$

for all large R.

Remark. Lagarias / Pleasants '03 bring forward both linearly repetitive and densely repetitive sets as 'perfectly ordered' aperiodic sets.

Relationship between densely repetitive and linearly repetitive sets.

Theorem (L. ’04). *Let Λ be a linearly repetitive FLC-Delone set, which does not admit a nontrivial period. Then, Λ is densely repetitive.*

Remark. Solves a conjecture in Lagarias / Pleasants ’03.

Complexity of aperiodic linearly repetitive Delone sets.

Proof amounts to showing

$$p_{\Lambda}(R) \geq \kappa \cdot R^N$$

for linearly repetitive FLC-Delone sets (without periods). This has following consequence.

Corollary. *Let Λ be a linearly repetitive FLC-Delone set, which does admit a non-trivial period. Then, there exist $\kappa, \lambda > 0$ with*

$$\kappa \cdot R^N \leq p_{\Lambda}(R) \leq \lambda \cdot R^N$$

for all $R \geq 1$.

Remark. Compare Durand '00 for subshift version.

Patch counting entropy.

For an FLC-Delone set one defines the *patch counting entropy* by

$$h_{pc}(\varLambda) := \lim_{R\to\infty} \frac{\log p_{\varLambda}(R)}{|B_R(0)|}.$$

(Existence follows from suitable use of subadditivity.)

Corollary. *Any linearly repetitive Delone set has* $h_{pc}(\varLambda) = 0$*.*

I.8. Uniform patch frequencies

Λ FLC-Delone set has *uniform patch frequencies* if for any patch P of size R the limit

$$\lim_{S \to \infty} \frac{\sharp\{x \in B_S(a) \cap \Lambda : (\Lambda - x) \cap B_R(x) = P\}}{|B_S(a)|}$$

exists uniformly in $a \in \mathbb{R}^N$.

Remark. For lattices this holds. In fact, there the limit is always the density of the lattice.

Remark. Both repetitivity and uniform patch frequency can be seen as notions or order giving precise version of 'regular distribution' of patches in Λ.

Repetitivity is a topological notion. Uniform patch frequency is a statistical notion.

They are independent of each other.

Examples for uniform patch frequencies.

Main results of Lagarias / Pleasants ’03 show that both

- linearly repetitive,

- densely repetitive

FLC-Delone sets have uniform patch-frequencies.

Remarks on averaging for subadditive sequences:

(a) As shown by Damanik / Lenz '04 linearly repetitive Delone sets even allow for a subadditive ergodic theorem (which implies uniform patch frequencies).

(b) As shown by Besbes / Boshernitzan / Lenz '12 one can even characterize linear repetitivity by validity of a subadditive ergodic theorem (together with 'roundness' condition on Voronoi cells).

Remarks on averaging for subadditive sequences - symbolic case:

Let ω be repetitive sequence over a finite alphabet be given and $\mathcal{L}(\omega)$ the associated language.

$F : \mathcal{L}(\omega) \longrightarrow \mathbb{R}$ *subadditive* if

$$F(xy) \leq F(x) + F(y),$$

whenever $xy \in \mathcal{L}(\omega)$.

ω satisfies *positivity of weights* if there exists a $c > 0$ with

$$\lim_{|x|\to\infty} \frac{\sharp_v(x)|v|}{|x|} \geq c$$

for all $v \in \mathcal{L}(\omega)$.

Theorem. *The following assertions are equivalent:*

(i) For any subadditive F the limit

$$\lim_{|x|\to\infty} \frac{F(x)}{|x|}$$

exists.

(ii) ω satisfies positivity of weights.

(iii) ω is linearly recurrent.

Proof of (i)$\Longleftrightarrow$ (ii) due to L. '01, Proof of (ii)$\Longleftrightarrow$ (iii) due to Boshernitzan 90ies (unpublished).

I.9. Meyer-Moody-theory and a result of Lagarias

We now turn to a class of Delone sets with properties very similar to lattices. The corresponding theory goes back to Meyer '73 and was then further developed by Moody '96 and Schlottmann '00 (and many more....)

Theorem (Meyer '73)**.** *Let $\Lambda \subset \mathbb{R}^N$ be an FLC-Delone set. Then, the following assertions are equivalent:*

(i) $\Lambda - \Lambda$ is uniformly discrete.

(ii) For any $\varepsilon > 0$ the set

$$\{k \in \mathbb{R}^N : |e^{2\pi i k x} - 1| \leq \varepsilon \text{ for all } x \in \Lambda \}$$

is relatively dense.

Such Delone sets are nowadays known as *Meyer sets*.

Remarks.

Compare lattices!

FLC means that $\Lambda - \Lambda$ is locally finite. The Meyer property is a substantial strengthening of FLC.

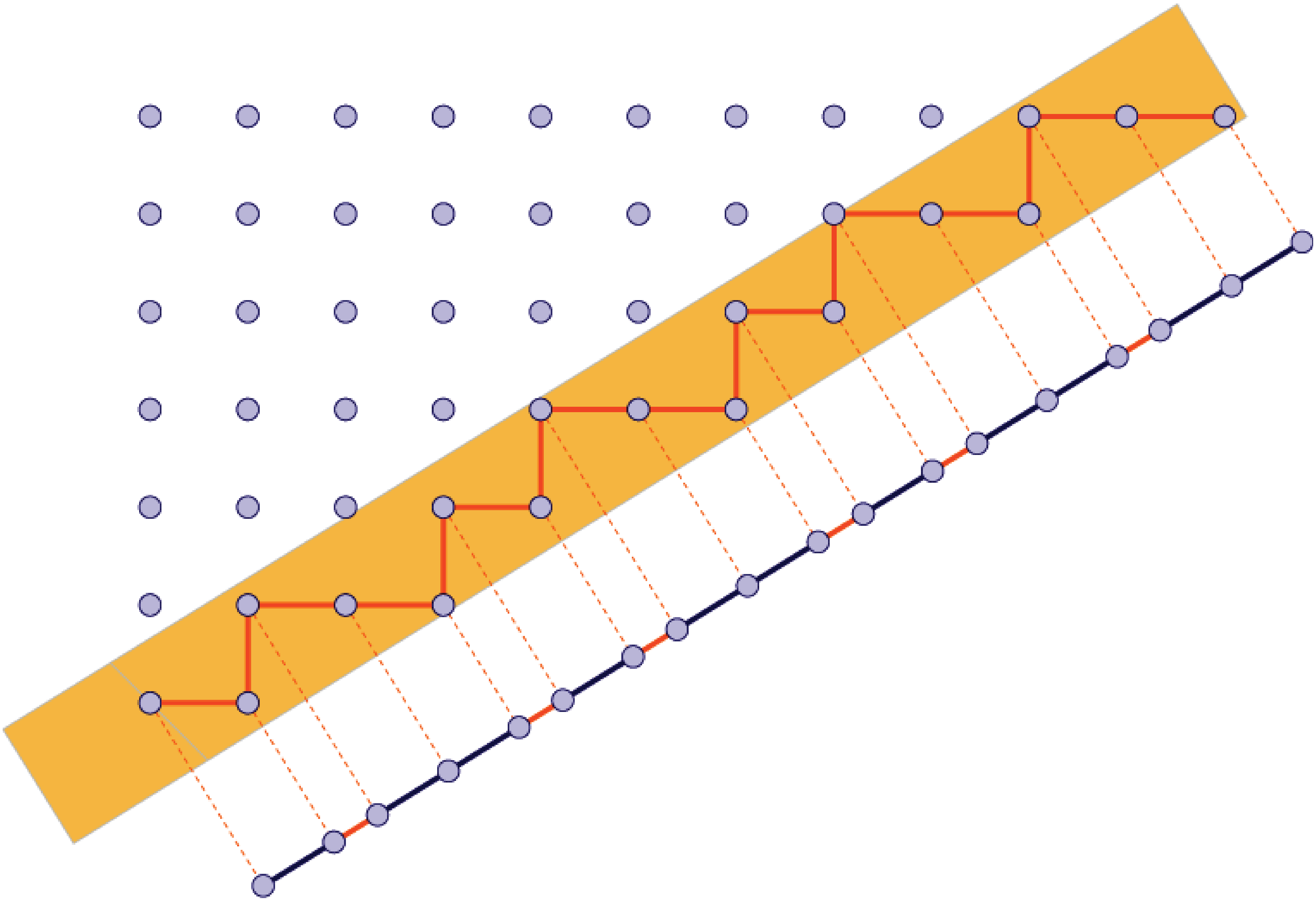
The Meyer property is very strong (as can be seen from both conditions in the theorem).

As discussed above FLC is not an order requirement (as typically tilings with finitely many tiles will have this property).

The Meyer property is an order requirement. A 'typical' FLC set will not be Meyer: Take e.g. two intervals with length 1 and α irrational and consider a generic tiling of $\mathbb{R}$ with these tiles.

The previous considerations give two equivalent ways in which Meyer sets are generalizations of lattices.

Now, we are heading for yet another such characterization: Meyer sets are 'shadows of lattices' (in the platonic sense).



Cut and project schemes and model sets. A *cut and project scheme* $(\mathbb{R}^N, H, \widetilde{L})$ is given as follows:

$$
\begin{array}{ccccc}
\mathbb{R}^N & \xleftarrow{\ \pi\ } & \mathbb{R}^N \times H & \xrightarrow{\ \pi_{\text{int}}\ } & H \\
\cup & & \cup & & \cup \text{dense} \\
L & \xleftarrow{\ 1-1\ } & \widetilde{L} & \longrightarrow & L^\star \\
\| & & & & \| \\
L & & \xrightarrow{\qquad \star \qquad} & & L^\star
\end{array}
$$

where

- H is a locally compact, σ-compact group, called the *internal space*,
- $\tilde{L}$ is a *lattice* in $\mathbb{R}^N \times H$,
- π and π_{int} are the canonical projections.
- π is one-to-one and π_{int} has dense range.

Then, $L := \pi(\widetilde{L})$ and $L^\star := \pi_{\text{int}}(\widetilde{L})$ are groups. As π is one-to-one, there is a uniquely defined group homomorphism

$$\star : L \longrightarrow L^\star$$

such that $(x, h) \in \widetilde{L}$ if and only if $h = x^\star$.

Given an cut and project scheme $(\mathbb{R}^N, H, \widetilde{L})$ we define for a so called window $W \subset H$ the associated subset of $\mathbb{R}^N$ by

$$人(W) \; := \; \{x \in L : x^\star \in W\}.$$

Example. Fibonacci chain.

Example. Penrose tiling.

Let a cut and project scheme $(\mathbb{R}^N, H, \widetilde{L})$ be given and

$$\curlywedge(W) \; := \; \{x \in L : x^\star \in W\}.$$

Proposition. *$\emptyset \neq V \subset H$ open. Then, $\curlywedge(V)$ is relatively dense.*

Proposition. *$K \subset H$ compact. Then, $\curlywedge(K)$ is uniformly discrete.*

Theorem. *Let $\varLambda = \curlywedge(W)$ for a compact W with non-empty interion. Then, $\varLambda$ is uniformly discrete and relatively dense. Moreover, $\varLambda - \varLambda$ is uniformly discrete as well. In particular, $\varLambda$ is Meyer (and, hence, also FLC).*

Meyer sets and cut and project schemes.

Here comes the characterization of Meyer sets via cut and project schemes.

Theorem (Meyer '73, Moody '96)**.** *Let* Λ *be a Delone set. Then,* Λ *is a Meyer set if and only if there exists a cut and project scheme and a compact* $W \subset H$ *with* $\Lambda \subset ⅄(W)$.

The previous considerations give three equivalent ways in which Meyer sets are generalizations of lattices.

Now, we are heading for yet another such characterization:

A result of Lagarias.

Recall that a subset Γ of $\mathbb{R}^N$ is a group if and only if $\Gamma - \Gamma \subset \Gamma$.

Theorem (Lagarias '99)**.** *Let Λ be an FLC-Delone set. Then, Λ is a Meyer set if and only if*

$$\Lambda - \Lambda \subset \Lambda + F$$

for some finite F.

Remark. The results says

$$\bigcup_{x\in\Lambda}(\Lambda - x) \subset \bigcup_{y\in F}(\Lambda + y)$$

for Meyer sets.

Summary. Meyer sets have many claims to be considered as generalized lattices:

Theorem. *Let Λ be an FLC-Delone set. Then, the following assertions are equivalent:*

(i) $\Lambda - \Lambda$ *is uniformly discrete.*

(ii) $\Lambda - \Lambda \subset \Lambda + F$ *for a finite set* F.

(iii) Λ *is harmonious i.e.*

$$\{k : |e^{2\pi ikk} - 1| \leq \varepsilon \text{ for all } x \in \Lambda\}$$

is relatively dense for any $\varepsilon > 0$.

(iv) Λ *comes from a cut and project scheme.*

II. Dynamical systems and the hull of an FLC-Delone set

1. The torus.
2. The hull and its topology.
3. Characterizing notions of order via dynamical systems.
4. Continuous eigenfunctions and the MEF.
5. Meyer sets and the Kellendonk-Sadun theorem.
6. The hierarchy of Meyer dynamical systems.

II.1. The torus

Γ lattice in $\mathbb{R}^N$ gives rise to *torus*

$$\mathbb{T} := \mathbb{R}^N/\Gamma.$$

There is a canonical map $j : \mathbb{R}^N \longrightarrow \mathbb{T}, j(t) := t + \Gamma$.

This induces an action of $\mathbb{R}^N$ on $\mathbb{T}$ via

$$\mathbb{R}^N \times \mathbb{T} \longrightarrow \mathbb{T}, (t, \xi) \mapsto j(t) + \xi.$$

The map j is onto. Thus, this action is minimal and uniquely ergodic (i.e. each orbit is dense and there is a unique invariant probability measure on $\mathbb{T}$ viz Haar measure).

II.2. The hull and its topology

Consider an FLC-Delone set Λ. The *hull* of Λ is defined as

$$\Omega(\Lambda) := \overline{\{t + \Lambda : t \in \mathbb{R}^N\}},$$

where the closure is taken in a certain topology. This topology can be defined in various ways:

- via a stereographic projection.
- via weak topology on measures.
- via a metric: Λ_1 and Λ_2 are close if they agree on a large ball after a small translation.

,

Theorem. *The hull $\Omega(\Lambda)$ is compact and*

$$\alpha : \mathbb{R}^N \times \Omega(\Lambda) \longrightarrow \Omega(\Lambda), (t, \Lambda) \mapsto t + \Lambda,$$

is a continuous action of $\mathbb{R}^N$ on Λ.

By the theorem $(\Omega(\Lambda), \mathbb{R}^N)$ is a topological dynamical system.

Remark. This approach to tilings (Delone sets) via hulls goes (at least) back to work of Radin '90 and Radin / Wolf '91; see also Rudolph.

II.3. Characterizing notions of order via dynamical systems

Recall: A dynamical system $(\Omega, \mathbb{R}^N)$ is

- *minimal* if each orbit is dense,

- uniquely ergodic if there exists only one invariant probability measure on Ω.