# Diffraction spectrum of a Rudin–Shapiro-like sequence

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Transversal aspects of tilings
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# Motivation

In 1950, R. Salem asked the following question related to Fourier/Harmonic analysis:

Does there exist a sequence of $\epsilon_n \in \pm 1$ such that

$$\sup_{\theta \in \mathbb{R}} \left| \sum_{n<N} \epsilon_n e(n\theta) \right| \leq C\sqrt{N},$$

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# Different representations of the Rudin–Shapiro sequence

$r_n := (-1)^{e_{2;11}(n)}$, where $e_{2;11}(n)$ counts the number of (possibly overlapping) occurrences of the block 11 in the binary representation of $n$.

**Example**

- $0_2 = 0, \quad e_{2;11}(0) = 0, \quad r_0 = +1;$
- $1_2 = 1, \quad e_{2;11}(1) = 0, \quad r_1 = +1;$
- $2_2 = 10, \quad e_{2;11}(2) = 0, \quad r_2 = +1;$
- $3_2 = 11, \quad e_{2;11}(3) = 1, \quad r_3 = -1.$

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# Dynamical representation

Iterating the following map:

$$\varrho_{\mathrm{RS}} : \begin{array}{l} 0 \mapsto 02 \\ 1 \mapsto 32 \\ 2 \mapsto 01 \\ 3 \mapsto 31 \end{array} .$$

- Primitivity: When there exists some $k \in \mathbb{N}$ such that every $a_j$ is a subword of each $\varrho^k(a_i)$.
- Legality: A finite word is called *legal* if it occurs as a subword of $\varrho^k(a_i)$ for some $1 \leq i \leq n$ and some $k \in \mathbb{N}$.

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# What is known?

Let $S(N) := \sum_{0 \le n \le N} r_n$.

**Theorem (Allouche, Shallit)**

$S(N) = \sqrt{N} G(\log_4 N)$, where $G$ is a certain function that oscillates periodically between $\sqrt{3}/3$ and $\sqrt{2}$.

**Proposition (Baake,Grimm)**

The Rudin–Shapiro sequence has purely absolute continuous diffraction spectrum.

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# A Rudin–Shapiro-like sequence

$(i_n)_{n\geq 0}$, defined by $i_n = (-1)^{\mathrm{inv}_2(n)}$, where $inv_2(n)$ counts the number of inversions (occurrences of 10 as a scattered subsequence) in the binary representation of $n$.

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**Proposition (Yee, Lafrance, Rampersad)**

The sequence $(i_n)_{n\geq 0}$ satisfies the following recurrence relations:

$$\begin{aligned} i_{4n} &= i_n \\ i_{4n+1} &= i_{2n} \\ i_{4n+2} &= -i_{2n} \\ i_{4n+3} &= i_n. \end{aligned}$$

# What is not known?

**Question 1.**

What is the diffraction spectrum of this sequence?

**Question 2. (Yee, Lafrance, Rampersad)**

Does it satisfy $\sup_{\theta\in\mathbb{R}} \left|\sum_{n<N} i_n e(n\theta)\right| \leq C\sqrt{N}$?

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# Mathematical diffraction theory

Autocorrelation measure:

$$\gamma = \sum_{m \in \mathbb{Z}} \eta(m)\delta_m$$

Autocorrelation coefficient:

$$\eta(m) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} v(i)v(i+m)$$

Diffraction measure $\widehat{\gamma}$:

- Fourier transform of the autocorrelation measure.

**Theorem (Lebesgue decomposition theorem)**

$$\widehat{\gamma} = \widehat{\gamma_{\mathrm{pp}}} + \widehat{\gamma_{\mathrm{sc}}} + \widehat{\gamma_{\mathrm{ac}}}.$$

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If a sequence satisfy $\sup_{\theta\in\mathbb{R}} \left|\sum_{n<N} \epsilon_n e(n\theta)\right| \le C\sqrt{N}$, then it has purely absolute continuous diffraction spectrum.

**Proof:**

**Step 1**: Define the correlation function:

$$\eta(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \epsilon(n+k)\epsilon(n),$$

where $N \in \mathbb{Z}$ and for every $k \in \mathbb{N}$.

By Heglotz-Bochner theorem, $\eta$ is the Fourier transform of a position measure $\sigma$ on $[0, 1)$, which we call a correlation measure.

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# proof

**Step 2:**

**Theorem (Baake, Grimm)**

Let $\varrho$ be a primitive substitution on a finite alphabet. Its hull $\mathbf{X}(w) := \overline{\{S^i(w) : i \in \mathbb{Z}\}}$ is then *strictly ergodic* under the $\mathbb{Z}$-action of the shift.

**Strict ergodicity=unique ergodicity+minimality.**

**Step 3:**

**Proposition (Queffélec)**

If $\sigma$ is the unique correlation measure of the sequence $\gamma$, $\sigma$ is the weak-$*$ limit point of the sequence of absolute continuous measures $R_N \cdot m$, where $m$ is the Haar measure and $R_N = \frac{1}{N}\left|\sum_{n<N} \epsilon_n e(n\theta)\right|^2$.

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# Last step of the proof

Denote $\zeta_N = R_N \cdot m$ and suppose $\zeta_N$ converges weak-$*$ to $\zeta$.

Take a function $g \in C_c(\mathbb{R}^d)$, a continuous complex-valued function with compact support.

$$\zeta_N(g) = \int g \cdot \frac{1}{N} \left| \sum_{n<N} \epsilon_n e(n\theta) \right|^2 dm,$$

We obtain $\zeta(g) \leq C \int g \; dm$, this implies absolute continuity.

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# Thank you for your attention!