FREEZING PHASE TRANSITION WITH SUPPORT IN A QUASICRYSTAL IN ERGODIC THEORY

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1. Curie-Weiss model with respect to Ergodic theory

1.1. Different but close settings.

1.1.1. *The Curie-Weiss model. Probabilistic setting 1.* We consider the sets $\Lambda = \{-1,+1\}$ and $\Sigma := \Lambda^{\mathbb{N}}$. A point x in Σ is a sequence $x_0, x_1, \dots$ (also called an infinite word) where the x_i are in Λ. Most of the times we shall use the notation $x = x_0x_1x_2\dots$. A $x_i \in \Lambda$ can either be called a letter, or a digit or a symbol.

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If $\omega_0 \dots \omega_{n-1}$ is a finite word, we set

$$H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} \omega_j \omega_i. \tag{1}$$

It is called the *Curie-Weiss Hamiltonian*. The *empirical magnetization* for ω is $m_n(\omega) := \dfrac{1}{n} \displaystyle\sum_{j=0}^{n-1} \omega_j$. Then we have $H_n(\omega) = -\frac{n}{2}(m_n(\omega))^2$.

We denote by $\mathbb{P} := \rho^{\otimes \mathbb{N}}$ the product measure on Σ_2, where ρ is the uniform measure on $\{-1, 1\}$, i.e. $\rho(\{1\}) = \rho(\{-1\}) = \frac{1}{2}$, and we define the *probabilistic Gibbs measure* (PGM for short) $\mu_{n,\beta}$ on Σ_2 by

$$\mu_{n,\beta}(d\omega) := \frac{e^{-\beta.H_n(\omega)}}{Z_{n,\beta}} \mathbb{P}(d\omega), \tag{2}$$

where $Z_{n,\beta}$ is the normalization factor

$$Z_{n,\beta} = \frac{1}{2^n} \sum_{\omega',\ |\omega'|=n} e^{-\beta.H_n(\omega')}.$$

Note that $\mu_{n,\beta}$ can also be viewed as a probability defined on Λ^n.

1.1.2. *Ergodic and Dynamical settings.* The shift map σ is defined on Σ by

$$\sigma(x_0 x_1 x_2 \dots) = x_1 x_2 \dots.$$

The distance between two points $x = x_0 x_1 \dots$ and $y = y_0 y_1 \dots$ is given by

$$d(x, y) = \frac{1}{2^{\min\{n,\ x_n \neq y_n\}}}.$$

We sometimes represent this distance graphically as follows:

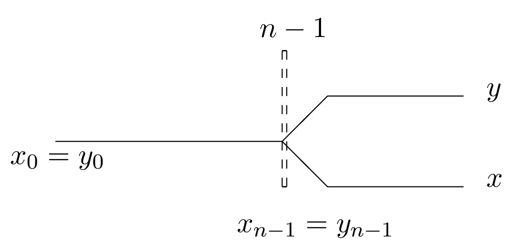

FIGURE 1. The sequence x and y coincide for digits 0 up to $n-1$ and then split.

A finite string of symbols $x_0 \dots x_{n-1}$ is also called a *word*, of length n. For a word w, its length is $|w|$. A *cylinder* (of length n) is denoted by $[x_0 \dots x_{n-1}]$. It is the set of points y such that $y_i = x_i$ for $i = 0, \dots n-1$.

For $\phi : \Sigma \to \mathbb{R}$ continuous and $\beta > 0$, the *pressure function* is defined by

$$\mathcal{P}(\beta.\phi) := \sup_{\mu} \left\{ h_\mu + \beta \int \phi \, d\mu \right\}, \tag{3}$$

where the supremum is taken among the set $\mathcal{M}_\sigma(\Sigma_2)$ of σ-invariant probabilities on Σ and h_μ is the Kolmogorov-Sinaï entropy of μ. The real parameter β is assumed to be positive because it represents the inverse of the temperature in statistical mechanics. It is known that the supremum is actually a maximum and any measure for which the maximum is attained in (3) is called an *equilibrium state for* $\beta.\phi$. We refer the reader to [3, 19] for basic notions on thermodynamic formalism in ergodic theory.

If ϕ is Lipschitz continuous then the Ruelle-Griffith theorem (see [11]) states that for every β, there is a unique equilibrium state for $\beta.\phi$, which is denoted by $\widetilde{\mu}_\beta$ if the choice of ϕ is clear. The measure $\widetilde{\mu}_\beta$ is *ergodic* and it shall be called the *dynamical Gibbs measure* (DGM for short). It satisfies for every $x = x_0 x_1 \ldots$ and for every n

$$e^{-C_\beta} \leq \frac{\widetilde{\mu}_\beta([x_0 \ldots x_{n-1}])}{e^{S_n(\phi)(x) - n\mathcal{P}(\beta.\phi)}} \leq e^{C_\beta}, \tag{4}$$

where C_β is a positive real number depending only on β and ϕ (but not on x or n), and $S_n(\phi)$ stands for $\phi + \phi \circ \sigma + \ldots + \phi \circ \sigma^{n-1}$.

1.1.3. *Relations between these definitions of Gibbs measures.* Our main question is to understand relations/differences between these two definitions of Gibbs measures. As we pointed out above, $\mu_{n,\beta}$ lives in Λ^n whereas $\widetilde{\mu}_\beta$ lives in $\Sigma = \Lambda^{\mathbb{N}}$.

If P_n, P are probability measures on the Borel sets of a metric space S, we say that P_n converges weakly to P if $\int_S f \, dP_n \to \int_S f \, dP$ for each f in the class $C_b(S)$ of bounded, continuous real functions f on S. In this case we write $P_n \underset{n\to+\infty}{\xrightarrow{w}} P$.

Then, our first result concerns the weak convergence of the measures $\mu_{n\beta}$.

Theorem 1. Weak convergence

Let ξ_β be the unique point in $[0,1]$ which realizes the maximum for $\varphi_I(x) := \log(\cosh(\beta.x)) - \frac{\beta}{2}x^2$. Let $\widetilde{\mu}_b^+$ and $\widetilde{\mu}_b^-$ be the dynamical Gibbs measures for $b.\mathbb{1}_{[+1]}$ and $b.\mathbb{1}_{[-1]}$ respectively. Then

$$\mu_{n,\beta} \underset{n\to+\infty}{\xrightarrow{w}} \begin{cases} \widetilde{\mu}_0 & \text{if } \beta \leq 1, \\ \dfrac{1}{2}\left[\widetilde{\mu}^+_{2\beta.\xi_\beta} + \widetilde{\mu}^-_{2\beta.\xi_\beta}\right] & \text{if } \beta > 1. \end{cases} \tag{5}$$

Remark 1. *Actually $\mu_{n,\beta}$ converges towards $\frac{1}{2}\left[\widetilde{\mu}^+_{2\beta.\xi_\beta} + \widetilde{\mu}^-_{2\beta.\xi_\beta}\right]$ for every $\beta > 0$ since we shall see that for $\beta \leq 1$ we have $\xi_\beta = 0$, and it is clear that $\widetilde{\mu}^+_0 = \widetilde{\mu}^-_0 = \widetilde{\mu}_0 = \rho^{\otimes\mathbb{N}}$.*

We refer to [8], sections IV.4 and V.9, for discussion of the Curie-Weiss model and historical references. By using this theorem, Orey ([16], Corollary 1.2) proved by a nice simple probabilistic argument the weak convergence of $\mu_{n,\beta}$ towards an explicit atomic measure.

Remark 2. It is said that there is a phase transition at $\beta = 1$.

1.2. **Proof of Theorem 1.** To prove the convergence of $\mu_{n,\beta}$ towards μ, it is enough to show that for every cylinder C,

$$\lim_{n\to\infty} \mu_{n,\beta}(C) = \mu(C). \tag{6}$$

First we justify that φ_I admits a unique maximum in $[0,1]$ and use this point to get convergence for $\mu_{n,\beta}(C)$, where C is any cylinder. In the second subsection we show that this limit is equal to the right convex combinations of DGM's.

1.2.1. *The auxiliary function φ_I and limit for $\mu_{n,\beta}$.* We recall that we set $\varphi_I(x) := \log(\cosh(\beta x)) - \frac{\beta}{2}x^2$.

Lemma 1.1. Maxima for φ_I

The function φ_I attains its maximum on $\mathbb{R}^+$ at a unique point ξ_β which is the unique positive solution of the equation $\tanh(\beta x) = x$. *If $\beta \leq 1$, then $\xi_\beta = 0$.*

Proof. Note that $\varphi_I'(x) = \beta\left(\tanh(\beta.x) - x\right)$ and $\varphi_I''(x) = \beta.(\beta - 1 - \beta.\tanh^2(\beta.x))$. If $\beta \leq 1$, φ_I'' is non-positive, thus φ_I' decreases and $\varphi_I'(0) = 0$ yields that φ_I is a decreasing function. The maximum is then attained for $\xi_\beta = 0$.

If $\beta > 1$, then φ_I'' is positive and then negative, which yields that φ_I' is first an increasing and then a decreasing function. Note that $\varphi_I'(0) = 0$ and $\varphi_I'(1) < 0$, which shows that φ_I' is positive on some interval $]0, \xi_\beta[$ with $0 < \xi_\beta < 1$ and negative on $]\xi_\beta, +\infty[$. Consequently, φ_I reaches its maximal value on $\mathbb{R}$ at the points $\pm\xi_\beta$ defined by

$$\tanh(\beta.\xi_\beta) = \xi_\beta. \tag{7}$$

□

Now we are ready to compute the limit of a fixed cylinder. Let $\omega = \omega_0 \dots \omega_{p-1}$ be a finite word of length p. We denote by $S_p(\omega) = \sum_{i=0}^{p-1} \omega_i$ the sum of the p digits of ω.

Lemma 1.2. Limit of the measure of a fixed cylinder

$$\lim_{n\to\infty} \mu_{n,\beta}([\omega_0 \dots \omega_{p-1}]) = \begin{cases} \dfrac{1}{2^p} \ \textit{if } \beta \leq 1, \\ \dfrac{1}{2}\left(f(\xi_\beta) + f(-\xi_\beta)\right) \ \textit{if } \beta > 1, \end{cases} \tag{8}$$

where

$$f(y) = \frac{e^{\beta y S_p(\omega)}}{(e^{\beta y} + e^{-\beta y})^p}.$$

Proof. For big n,

$$[\omega]=\bigsqcup[\omega_0\ldots\omega_{p-1}\alpha_p\ldots\alpha_{n-1}]=\bigsqcup_{\alpha,\ |\alpha|=n-p}[\omega\alpha],$$

and

$$\mu_{n,\beta}([\omega])=\sum_{\alpha,\ |\alpha|=n-p}\mu_{n,\beta}([\omega\alpha])=\frac{\sum\limits_{\alpha,\ |\alpha|=n-p}e^{-\beta H_n(\omega\alpha)}}{\sum\limits_{\alpha,\ |\alpha|=n}e^{-\beta H_n(\alpha)}}.$$

We set $s:=\sum_{i=0}^{p-1}\omega_i$, $S(\alpha):=\sum_{i=p}^{n-1}\alpha_i$, and $S_n(\omega\alpha):=s+S(\alpha)$. Then,

$$H_n(\omega\alpha)=-\frac{1}{2n}(s+S(\alpha))^2.$$

We use the equality

$$e^{a^2}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}+\sqrt{2}ax}\,dx,$$

sometimes called the Hubbard-Stratonovich transformation ([12],[20]), to compute the following.

$$\begin{aligned}\sum_{\alpha,\ |\alpha|=n-p}e^{\frac{\beta}{2n}S_n^2(\omega\alpha)} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}\sum_{\alpha}e^{\sqrt{\frac{\beta}{n}}xS_n(\omega\alpha)}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}+\sqrt{\frac{\beta}{n}}xs}\sum_{\alpha}e^{\sqrt{\frac{\beta}{n}}xS(\alpha)}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}+\sqrt{\frac{\beta}{n}}xs}2^{n-p}\left(\cosh\left(\sqrt{\frac{\beta}{n}}x\right)\right)^{n-p}\,dx\\ &= \frac{2^{n-p}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}+\sqrt{\frac{\beta}{n}}xs+(n-p)\log\cosh\left(\sqrt{\frac{\beta}{n}}x\right)}\,dx.\end{aligned}$$

In this last integral we make the change of variable $\beta.y:=\sqrt{\frac{\beta}{n}}x$, and as

$$\exp\left(\beta.s.y-p\log\cosh\left(\beta.y\right)\right)=\frac{2^pe^{\beta yS_p(\omega)}}{(e^{\beta y}+e^{-\beta y})^p}$$

we obtain

$$\sum_{\alpha,\ |\alpha|=n-p}e^{\frac{\beta}{2n}S_n^2(\omega\alpha)}=\frac{2^n\sqrt{n\beta}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{n\varphi_I(y)}f(y)\,dy. \tag{9}$$

Similarly, $p=s=0$ yields

$$\sum_{\alpha,\ |\alpha|=n}e^{\frac{\beta}{2n}S_n^2(\alpha)}=\frac{2^n\sqrt{n\beta}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{n\varphi_I(y)}\,dy,$$

therefore we obtain that

$$\mu_{n,\beta}([\omega]) = \frac{\int_{-\infty}^{+\infty} e^{n\varphi_I(y)} f(y)\, dy}{\int_{-\infty}^{+\infty} e^{n\varphi_I(y)}\, dy}. \tag{10}$$

We recall that the Laplace method shows that if φ_I' vanishes on a single point ξ in the interior of the interval I and if $\varphi_I''(\xi) < 0$ and $f(\xi) \neq 0$, then

$$\int_I e^{n\varphi_I(y)} f(y) dy \sim_{n\to\infty} \frac{\sqrt{2\pi}}{\sqrt{|\varphi_I''(\xi)|}} e^{n\varphi_I(\xi)} f(\xi) n^{-1/2}.$$

If $\beta < 1$**:** we may consider $I = \mathbb{R}$ and $\xi = \xi_\beta = 0$. We thus get $\varphi_I''(0)) = \beta(\beta - 1)$, $f(0) = \frac{1}{2^p}$, and

$$\mu_{n,\beta}([\omega]) \sim_{n\to\infty} \frac{f(0)}{1} = \frac{1}{2^p}.$$

If $\beta = 1$**:** in this case $\varphi_I''(0)) = 0$ but the Laplace method still works if we consider the least integer k such that $\varphi_I^{(k)}(0) \neq 0$. We do not need to calculate it because we have as in the preceding case that

$$\int_{\mathbb{R}} e^{n\varphi_I(y)} f(y) dy \sim_{n\to\infty} \frac{1}{2^p} \int_{\mathbb{R}} e^{n\varphi_I(y)} dy,$$

therefore we still have $\mu_{n,\beta}([\omega]) \sim_{n\to+\infty} \dfrac{1}{2^p}$.

If $\beta > 1$**:** we may consider two intervals $\mathbb{R}_-$ and $\mathbb{R}_+$ and $\xi = \pm\xi_\beta$. Then we get

$$\int_{-\infty}^{+\infty} e^{n\varphi_I(y)} f(y) dy \sim_{n\to\infty} \frac{\sqrt{2\pi}}{\sqrt{|\varphi_I''(\xi_\beta)|}}\, e^{n\varphi_I(\xi_\beta)} \left(f(\xi_\beta) + f(-\xi_\beta)\right) n^{-1/2},$$

which yields

$$\mu_{n,\beta}([\omega]) \sim_{n\to\infty} \frac{f(\xi_\beta) + f(-\xi_\beta)}{2}. \qquad \square$$

1.2.2. *Identification of the limit as a convex combination of DGM's.* First, we point out that Lemma 1.2 yields $\lim\limits_{n\to+\infty} \mu_{n,\beta} = \widetilde{\mu}_0$ if $\beta \leq 1$. We thus have to deal with the case $\beta > 1$.

Let us first compute the dynamical Gibbs measures $\widetilde{\mu}_b^+$ and $\widetilde{\mu}_b^-$. We denote by $L_{p,+}(\omega) := \sum_{k=0}^{p-1} \mathbf{1}_{\omega_k=1}$ the number of digits of ω which take the value 1, and similarly $L_{p,-}(\omega) := \sum_{k=0}^{p-1} \mathbf{1}_{\omega_k=-1}$ is the number of digits which take the value -1.

Lemma 1.3. Computation for $\widetilde{\mu}_b^\pm$

$$\widetilde{\mu}_b^+([\omega]) = \frac{e^{bL_{p,+}(\omega)}}{(e^b+1)^p} \quad \textit{and} \quad \widetilde{\mu}_b^-([\omega]) = \frac{e^{bL_{p,-}(\omega)}}{(e^b+1)^p}. \tag{11}$$

Proof. The function $b.\mathbb{1}_{[+1]}$ depends only on the zero coordinate. It is shown for instance in Example 4.2.2 of [13] that in this case the supremum in (3) is attained for the product measure $\mathbb{P}_p := \rho_p^{\otimes \mathbb{N}}$, where $p \in [0,1]$, $\rho_p = p\delta_1 + (1-p)\delta_{-1}$, and p satisfies

$$-p\log p - (1-p)\log(1-p) + bp = \sup_{q\in[0,1]} \left(-q\log q - (1-q)\log(1-q) + bq\right).$$

It is easy to show that $p = \frac{e^b}{1+e^b}$, and then

$$\widetilde{\mu}_b^+([\omega]) = \frac{e^{bL_{p,+}(\omega)}}{(e^b+1)^p}.$$

Exchanging the roles of $+1$ and -1 gives the equality

$$\widetilde{\mu}_b^-([\omega]) = \frac{e^{bL_{p,-}(\omega)}}{(e^b+1)^p}. \qquad \square$$

We recall that $S_p(\omega) = \sum_{k=0}^{p-1}\omega_k = L_{p,+}(\omega) - L_{p,-}(\omega) = 2L_{p,+}(\omega) - p = p - 2L_{p,-}(\omega)$. Then

$$\begin{aligned} f(\xi_\beta) &= \frac{e^{\beta\xi_\beta S_p(\omega)}}{(e^{\beta\xi_\beta} + e^{-\beta\xi_\beta})^p} \\ &= \frac{e^{\beta\xi_\beta(2L_{p,+}(\omega)-p)}}{(e^{\beta\xi_\beta} + e^{-\beta.\xi_\beta})^p} \\ &= \frac{e^{2\beta\xi_\beta L_{p,+}(\omega)}}{(e^{2\beta\xi_\beta}+1)^p}. \end{aligned}$$

Similarly we have $f(-\xi_\beta) = \dfrac{e^{2\beta\xi_\beta L_{p,-}(\omega)}}{(e^{2\beta\xi_\beta}+1)^p}$ and replacing these values in (8) we get

$$\lim_{n\to+\infty} \mu_{n,\beta}([\omega]) = \frac{1}{2}(\widetilde{\mu}^+_{2\beta.\xi_\beta}([\omega]) + \widetilde{\mu}^-_{2\beta.\xi_\beta}([\omega])),$$

and the theorem is proved.

1.3. **A more general result for the Curie-Weiss-Potts model.** The *Curie-Weiss-Potts model* will be for $\Lambda = \{\theta^1, \dots, \theta^q\}$ with $q > 2$. In that case we shall write Σ_q instead of Σ.

The Curie-Weiss-Potts Hamiltonian is defined for a finite word $\omega = \omega_0 \cdots \omega_{n-1}$ by

$$H_n(\omega) := -\frac{1}{2n}\sum_{i,j=0}^{n-1} \mathbb{1}_{\omega_j=\omega_i}. \tag{12}$$

We define the vector $L_n(\omega) = (L_{n,1}(\omega), \cdots, L_{n,q}(\omega))$ where

$$L_{n,k}(\omega) = \sum_{i=0}^{n-1} \mathbb{1}_{\omega_i=\theta^k}$$

is the number of digits of ω which take the value θ^k, so that we can write

$$\sum_{i,j=0}^{n-1} \mathbb{1}_{\omega_j=\omega_i} = \sum_{k=1}^{q} \left(\sum_{i=0}^{n-1} \mathbf{1}_{\omega_i=\theta^k}\right)^2 = \|L_n(\omega)\|^2,$$

where $\|\cdot\|$ stands for the euclidean norm on $\mathbb{R}^q$.

We denote by $\mathbb{P} := \rho^{\otimes\mathbb{N}}$ the product measure on Σ_q, where ρ is the uniform measure on Λ, i.e. $\rho = \frac{1}{q}\sum_{k=1}^{q} \delta_{\theta^k}$, and we define the probabilistic Gibbs measure $\mu_{n,\beta}$ on Σ_q by

$$\mu_{n,\beta}(d\omega) := \frac{e^{-\beta.H_n(\omega)}}{Z_{n,\beta}}\mathbb{P}(d\omega) = \frac{e^{\frac{\beta}{2n}\|L_n(\omega)\|^2}}{Z_{n,\beta}}\mathbb{P}(d\omega), \tag{13}$$

where $Z_{n,\beta}$ is the normalization factor

$$Z_{n,\beta} = \frac{1}{q^n}\sum_{\omega',\ |\omega'|=n} e^{\frac{\beta}{2n}\|L_n(\omega')\|^2}.$$

Now we can state the analog of Theorem 1.

Theorem 2. Weak convergence for the CWP model

For $1 \le k \le q$, $b \in \mathbb{R}$, let $\widetilde{\mu}_b^k$ be the dynamical Gibbs measure for $b.\mathbb{1}_{[\theta^k]}$. Let $\beta_c = \frac{2(q-1)\log(q-1)}{q-2}$. For $0 < \beta < \beta_c$ set $s_\beta = 0$ and for $\beta \ge \beta_c$ let s_β be the largest solution of the equation

$$s = \frac{e^{\beta s} - 1}{e^{\beta s} + q - 1}. \tag{14}$$

Then,

$$\mu_{n,\beta} \xrightarrow[n\to+\infty]{w} \begin{cases} \rho^{\otimes\mathbb{N}} & \text{if } 0 < \beta < \beta_c, \\ \dfrac{1}{q}\displaystyle\sum_{k=1}^{q} \widetilde{\mu}^k_{\beta.s_\beta} & \text{if } \beta > \beta_c, \\ \dfrac{A\,\widetilde{\mu}_0^1 + B\,\sum_{k=1}^{q}\widetilde{\mu}^k_{\beta_c.s_{\beta_c}}}{A + qB} & \text{if } \beta = \beta_c, \end{cases} \tag{15}$$

with $A = \left(1 - \frac{\beta_c}{q(q-1)}\right)^{\frac{q-2}{2}}$ and $B = \left(1 - \frac{\beta_c}{q}\right)^{\frac{q-2}{2}}$.

Remark 3. *Actually $\mu_{n,\beta}$ converges towards $\frac{1}{q}\sum_{k=1}^{q}\widetilde{\mu}^k_{\beta.s_\beta}$ for every $\beta \ne \beta_c$ since $s_\beta = 0$ for $\beta < \beta_c$, and it is clear that $\widetilde{\mu}_0^k = \rho^{\otimes\mathbb{N}}$ for each $1 \le k \le q$.*

We refer to [9] for discussion of the Curie-Weiss-Potts model and historical references. Orey ([16], Theorem 4.4) mentions the weak convergence of $\mu_{n,\beta}$ towards an explicit atomic measure, but he makes a mistake concerning the case $\beta = \beta_c$, as pointed out in [9].

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2. Construction of dynamical (local) equilibrium states

2.1. Classical theory.

2.1.1. *the Transfer operator.* We consider a fixed α-Hölder potential $A : \Sigma \to \mathbb{R}$.

We recall that $A : \Sigma \to \mathbb{R}$ is said to be α-**Hölder**, $0 < \alpha < 1$, if there exists $C > 0$, such that, for all x, y we have $|\, A(x) - A(y) \,| \leq C\, d(x, y)^{\alpha}$.

For a fixed value α, we denote by $\mathcal{H}_\alpha$ the set of α-Hölder functions $A : \Sigma \to \mathbb{R}$. $\mathcal{H}_\alpha$ is a vector space.

For a fixed α, the norm we consider in the set $\mathcal{H}_\alpha$ of α-Hölder potentials A is

$$||A||_\alpha = \sup_{x \neq y} \frac{|A(x) - A(y)|}{d(x,y)^\alpha} + \sup_{x \in \Sigma} |A(x)|.$$

For a fixed α, the vector space $\mathcal{H}_\alpha$ is complete with the above norm.

Definition 2.1. *We denote by $\mathcal{L}_A : \mathcal{C}^0(\Sigma) \to \mathcal{C}^0(\Sigma)$ the Transfer operator corresponding to the potential A, which is given in the following way: for a given ϕ we will get another function $\mathcal{L}_A(\phi) = \varphi$, such that,*

$$\varphi(x) = \sum_{a,\ ax_0 \in \mathcal{A}} e^{A(ax)}\, \phi(ax).$$

In another form

$$\varphi(x) = \varphi(x_0 x_1 \cdots) = \sum_{a, ax_0 \in \mathcal{A}} e^{A(ax_0x_1x_2,\ldots)}\, \phi(ax_0x_1x_2\ldots).$$

The transfer operator is also called the Ruelle-Perron-Frobenius operator. It had been introduced by Ruelle and extends in some sense the matrices with positive entries. We remind that for such matrices, the Perron-Frobenius theorem gives information on the spectrum.

It is immediate to check that $\mathcal{L}_A$ acts on continuous functions. It also acts on α-Hölder functions if A is α-Hölder.

Consequently the dual operator acts on measures:

$$\begin{aligned} \mathcal{L}_A^* : \mu \ &\mapsto \ \nu \\ &\int \psi\, d\nu := \int \mathcal{L}_A(\psi)\, d\mu. \end{aligned}$$

Theorem 3 (see [3])**.** *Let λ_A be the spectral radius of $\mathcal{L}_A$. Then, λ_A is an eigenvalue for $\mathcal{L}_A$ and $\mathcal{L}_A^*$: there exists a probability measure ν_A such that*

$$\mathcal{L}_A^*(\nu_A) = \lambda_A \nu_A.$$

There exists a unique H_A, up to the normalization $\displaystyle\int H_A\,d\nu_A = 1$, such that

$$\mathcal{L}_A(H_A) = \lambda_A H_A.$$

The measure defined by $\mu_A = H_A \nu_A$ is σ-invariant and is the unique equilibrium state for A. The pressure satisfies $\mathcal{P}(1) = \lambda_A$

This measure is actually a *Gibbs* measure: there exists $C_A > 0$ such that for every $x = x_0 x_1 \ldots$ and for every n,

$$e^{-C_A} \leq \frac{\mu_A([x_0 \ldots x_{n-1}])}{e^{S_n(A)(x) - n\log\lambda_A}} \leq e^{C_A}. \tag{16}$$

These two inequalities yields that the free energy for μ_A is $\log\lambda_A$. The left-side inequality yields that for any other ergodic measure ν,

$$h_\nu + \int A\,d\nu < \log\lambda_A.$$

The same work can be done for $\beta.A$ instead of A. Actually the proof of Theorem 3 and general results for perturbations of spectrum of operators yield that $\beta \mapsto \mathcal{P}(\beta)$ is locally analytic. One argument of connexness shows that it is globally analytic.

2.1.2. *A complete and exact computation for one example.* Let us now assume that A depends on two coordinates, that is

$$A(x_0 x_1 x_2 \ldots) = A(x_0, x_1).$$

We denote by $A(i,j)$ the value of A in the cylinder $[ij]$, $i, j \in \{0,1\}$. In this case, the Transfer operator takes a simple form:

$$\mathcal{L}_A(\phi)(x_0 x_1 x_2 \ldots) = \sum_{a\in\{0,1\}} e^{A(ax_0)}\,\phi(ax_0 x_1 x_2 \ldots) = e^{A(0,x_0)}\phi(0x) + e^{A(1,x_0)}\phi(1x).$$

Let M be the matrix will all positive entries given by $M_{i,j} = e^{A(i,j)}$.

Lemma 2.2. *The spectral radius of $\mathcal{L}_A$ is also the spectral radius of M.*

Proof. Assume that ϕ is a function depending only on one coordinate, *i.e.*,

$$\phi(x_0 x_1 x_2 \ldots) = \phi(x_0).$$

Then, denote by abuse of notation ϕ the vector $(\phi(0), \phi(1))$. Then, for every j

$$\mathcal{L}_A(\phi)(j) = \sum_{j=0}^{1} M_{ij}.\phi(i),$$

which can be written as $\mathcal{L}_A(\phi) = M^*.\phi$. This yields that the spectral radius λ_M of M is lower or equal to λ_A.

We remind that the spectral radius is given by

$$\lambda_A := \limsup_{n\to+\infty} \frac{1}{n} \log |||\mathcal{L}_A^n|||, \text{ and } |||\mathcal{L}_A^n||| = \sup_{||\psi||=1} ||\mathcal{L}^n(\psi)||_\infty.$$

The operator $\mathcal{L}_A$ is positive and this shows that for every n, $|||\mathcal{L}_A^n||| = ||\mathcal{L}_A^n(\mathbb{1})||_\infty$. Now, $\mathbb{1}$ depends only on 1 coordinate, which then $\mathcal{L}_A^n(\mathbb{1}) = M^n(\mathbb{1})$. This yields $\lambda_A \leq \lambda_M$. □

Theorem 4. (Perron-Frobenius) *Let $B = (b_{ij})$ be a $d \times d$ matrix with positive entries. Then, the spectral radius of B, say λ, is a simple dominated eigenvalue. The associated eigenspace is generated by some "positive" vector $u = (u_1, \cdots, u_d)$ with $u_i > 0$.*

By Theorem 4 there exists an eigenvector say $\nu = (\nu_0, \nu_1)$ with positive entries for the matrix M associated to λ_A. We may assume $\nu_0 + \nu_1 = 1$.

Let us define the 2×2 matrix $P_A = P_A(i,j)$ with

$$P_A(i,j) = \frac{e^{A(i,j)}\nu_j}{\lambda_A \nu_i}.$$

Note that P_A is a line stochastic matrix:

$$P_A(0,0) + P_A(0,1) = 1 = P_A(1,0) + P_A(1,1).$$

Theorem 4 applied to the adjoint matrix M^* yields a left-eigenvector for M with positive entries $H = (h_0, h_1)$. Set $\mu_i = h_i.\nu_i$ and assume normalization $\mu_0 + \mu_1 = 1$.

$$\mu.P_A = \mu$$

and $\mu$is the invariant measure associated to the Markov chain with transition matrix P_A. Moreover we get

$$\mu_A([x_0 \dots x_{n-1}]) = \mu_{x_0} P_A(x_0, x_1) \dots P_A(x_{n-2}, x_{n-1}).$$

The exact computation yields $\mu_A([x_0 \dots x_{n-1}]) = h_{x_0} e^{S_n(A)(x) - n \log \lambda_A}.\nu_{x_{n-1}}$. Since ν and h have positive entries, this shows that μ_A is a Gibbs measure.

Things can be summarized as follows:

Let $M = (M_{ij})$ be the matrix with entries $e^{A(i,j)}$. Let $\mathbf{r} = (r_1, \dots r_d)$ be the right-eigenvector associated to λ with normalization $\sum r_i = 1$. Let $\mathbf{l} = (l_1, \dots, l_d)$ be the left-eigenvector for λ with renormalization $\sum l_i r_i = 1$. Then, $\mathbf{r}$ is the eigenmeasure ν_A and $\mathbf{l}$ is the density H_A. The Gibbs measure of the cylinder $[i_0 \dots i_{n-1}]$ is $\mu_A = ([i_0 \dots i_{n-1}]) = l_{i_0} e^{S_n(A)(x) - n \log \lambda_A} r_{i_{n-1}}$

Remark 4. *Doing this with $A := \beta.(\mathbb{1}_{[\pm 1]})$ one recovers the Dynamical Gibbs measures from Theorem 1.*

2.1.3. *Phase transition.*

Definition 2.3. *We say that there is a transition phase at β_0 if the pressure function $\mathcal{P}(\beta)$ is not analytic at β_0. We say that it is a* freezing phase transition *if $\mathcal{P}(\beta)$ is affine for every $\beta > \beta_c$.*

In case of a freezing phase transition, after the transition $\mathcal{P}(\beta)$ is of the form

$$\mathcal{P}(\beta) = h + \beta.a.$$

It is then easy to check that a satisfies: $a = \max \displaystyle\int A \, d\mu$. In ergodic theory, a measure satisfying this last property is said to be *A-maximizing*. In Statistical Mechanics it is a *ground state*. The quantity h is called the *residual entropy*. It corresponds to the maximum of the Kolmogorov entropy among all A-maximizing measures.

One question we are interested in is to know if we can get freezing phase transition with support in a quasi-crystal set after the transition.

Remark 5. *It is noteworthy that the number of equilibrium state is "almost" independent to the analyticity of the pressure function $\beta \mapsto \mathcal{P}(\beta)$. There are examples of systems with a phase transition despite uniqueness of the equilibrium state (the Manneville-Pomeau maps, see* [21]*). On the contrary, there are examples where the pressure is analytic on some interval but there are finitely many different equilibrium states (see* [15]*).*

2.2. **On the road to detect freezing phase transition.**

2.2.1. *Induced map.* We consider a cylinder $J = [w_J]$ in Σ. For x in J, the first return time is $\tau(x) = \min\{n \geq 1, \sigma^n(x) \in J\} \leq +\infty$. Then we consider the first return map $F(x) := \sigma^{\tau(x)}(x)$. This map is well defined if $\tau(x) < +\infty$. The main important point is that the inverse branches are well defined everywhere on J: if x belongs to J and if x' satisfies $F(x') = x$, hence we can write x' under the form

$$x' = wx.$$

Then, for every y in J, $y'=wy$ satisfies $F(y')=y$ and $\tau(y')=\tau(x')=|w|$.

It is also well known that if $\widehat{\mu}$ is a σ-invariant probability measure with $\widehat{\mu}(J)>0$, then, the conditional measure $\mu:=\dfrac{\widehat{\mu}(.\cap J)}{\widehat{\mu}(J)}$ is F-invariant. Conversely, if μ is a F-invariant probability measure and $\displaystyle\int\tau\,d\mu<+\infty$, then there exists a unique σ-invariant probability $\widehat{\mu}$ such that $\mu:=\dfrac{\widehat{\mu}(.\cap J)}{\widehat{\mu}(J)}$ holds (see [6]).

At that stage we have two different dynamical systems: (Σ,σ) and (J,F). The question is to know if studying thermodynamic formalism for one system yields information on the thermodynamic formalism for the other one. By the Abramov formula (see [17] p. 257-258) we get

$$\begin{array}{rcl}
h_{\widehat{\mu}}(f)+\displaystyle\int A\,d\widehat{\mu} & \leq & \mathcal{P}\text{ with equality iff }\widehat{\mu}=\text{equil. state}\\
 & \Updownarrow & \\
h_{\widehat{\mu}}(f)+\displaystyle\int A\,d\widehat{\mu}-\mathcal{P} & \leq & 0\text{ with equality iff }\widehat{\mu}=\text{equil. state}\\
 & \Updownarrow & \\
\widehat{\mu}(R)\left(h_{\mu}(F)+\displaystyle\int S_{\tau(.)}(A)-\mathcal{P}.\tau(.)\,d\mu\right) & \leq & 0\text{ with equality iff }\widehat{\mu}=\text{equil. state}\\
 & \Updownarrow & \\
h_{\mu}(F)+\displaystyle\int S_{\tau(.)}(A)-\mathcal{P}.\tau(.)\,d\mu & \leq & 0\text{ with equality iff }\widehat{\mu}=\text{equil. state}
\end{array}$$

This simple sequence of inequalities shows that the thermodynamic formalism for (Σ,σ) and potential A is related to the thermodynamic formalism for (J,F) and $S_{\tau(.)}(A)(.)$.

As we have seen above, the thermodynamic quantities come from the spectrum of the Transfer Operator.

2.2.2. *Inducing scheme and local thermodynamic formalism.* Let $A:\Sigma\to\mathbb{R}$ be some potential function and $J=[w_J]$ be a cylinder. Consider the first return map to J, with return time $\tau(x)=\min\{n\geq 1,\sigma^n(x)\in J\}$. Then we define, for each $\beta>0$ and $Z\in\mathbb{R}$, an induced transfer operator by:

$$\mathcal{L}_{Z,\beta}(g)(x)=\sum_{n\in\mathbb{N}}\sum_{\substack{\tau(y)=n\\ \sigma^n(y)=x}}e^{\beta.S_n(A)(y)-nZ}g(y)$$

where $S_N(A)(y) = \sum_{k=0}^{N-1} A \circ \sigma^k(y)$ and g is a continuous function from J to $\mathbb{R}$.

For a given function A, it is a power series in e^{-Z}.

Theorem 5 ([14]). *We have with the previous notations:*

- *For every $\beta \geq 0$, there exists a minimal $Z_c(\beta) \in \mathbb{R} \cup \{-\infty\}$ such that for every $Z > Z_c(\beta)$, $\mathcal{L}_{Z,\beta}$ acts on $\mathcal{C}^0(J)$. In particular, for every $Z > Z_c(\beta)$, for every $x \in J$ and for every $g \in \mathcal{C}^0(J)$, $\mathcal{L}_{Z,\beta}(g)(x)$ converges.*
- *$\mathcal{P}(\beta) \geq Z_c(\beta)$.*
- *Let $\lambda_{Z,\beta}$ be the spectral radius for $\mathcal{L}_{Z,\beta}$ and for $Z > Z_c(\beta)$. Then $Z \mapsto \log \lambda_{Z,\beta}$ is a decreasing function and we have three possible cases given by Figure 2.*

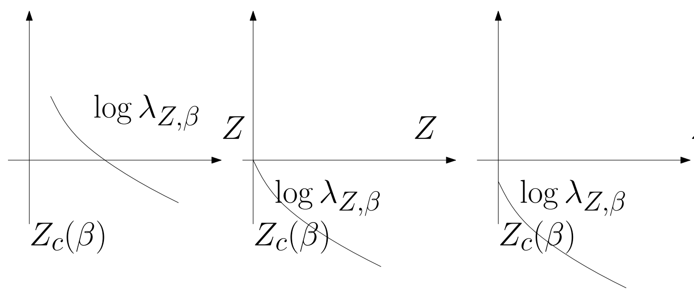

FIGURE 2. The tree possible graphs for $\log \lambda_{Z,\beta}$.

- *If case 1 holds, then $\log \lambda_{Z,\beta} = 0$ if and only if $Z = \mathcal{P}(\beta)$ and there is a unique equilibrium state for $\beta.A$; it is a fully supported measure in Σ. Moreover, $Z_c(\beta) < \mathcal{P}(\beta)$ and $\beta \to P(\beta)$ is analytic on the largest open interval where case 1 holds.*
- *If case 3 holds, then no equilibrium state gives positive weight to J.*

2.2.3. *Example: Hofbauer potential.* apply the method to the Hofbauer potential in $\{0,1\}^{\mathbb{N}}$

$$A(x) = \begin{cases} -\log(1+\frac{1}{n}) \text{ if } x = 0^n 1 \dots, \\ -\alpha < 0 \text{ if } x = 1 \dots. \end{cases}$$

This case is usually associated to the Manneville-Pomeau map, say *e.g.*

$$f : [0,1] \circlearrowleft \ x \mapsto \begin{cases} \dfrac{x}{1-x} \text{ if } x \in [0, \dfrac{1}{2}], \\ 2x \mod 1 \text{ if } x \in [\frac{1}{2}, 1]. \end{cases}$$

One can find in [1] a description of why these two cases are associated, and actually similar.

In that case we induce on the cylinder $J := [1]$. Note that only one orbit does not enter into $[1]$, and it is $0^\infty = 000\ldots$. Moreover, for any $x \in [1]$, and for every $\beta > 0$

$$\mathcal{L}_{Z,\beta}(1\!\!1_{[1]})(x) = e^{-\beta.\alpha} \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^{\beta} e^{-Z.(n+1)}.$$

For every $\beta \geq 0$, this series converges if $Z > 0$ and diverges for $Z < 0$. Therefore $Z_c = 0$, and we point-out that

$$0 = h_{\delta_{0^\infty}} + \beta. \int \phi \, d\delta_{0^\infty}.$$

Now, the form of the potential also yields $\lambda_{\beta,Z} = \mathcal{L}_{\beta,Z}(1\!\!1_{[1]})(x)$ for any x in $[1]$. Let us study the critical case $Z = Z_c$:

$$\lambda_{0,\beta} := e^{-\beta.\alpha} \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^{\beta}.$$

For $\beta \leq 1$, $\lambda_{0,\beta} = +\infty$. Furthermore, the function $\beta \mapsto \lambda_{0,\beta}$ is decreasing on $]1, +\infty[$, goes to $+\infty$ if $\beta \to 1$ and goes to 0 if $\beta \to +\infty$. Therefore, there exists a unique β_c such that $\lambda_{0,\beta_c} = 1$.

For $\beta > \beta_c$, no equilibrium state gives positive weight to $[1]$, which means that δ_{0^∞} is the unique equilibrium state and the pressure is 0.

For $\beta < \beta_c$, the map $Z \mapsto \lambda_{Z,\beta}$ is decreasing, and there is a unique $Z = \mathcal{P}(\beta) > 0$ such that

$$\lambda_{\mathcal{P}(\beta),\beta} = 1.$$

As $\mathcal{P}(\beta) > 0 = Z_c$, we are in the case 1, and the associated measure $\widehat{\mu}_{\mathcal{P}(\beta)}$ satisfies

$$h_{\widehat{\mu}_{\mathcal{P}(\beta)}}(\sigma) + \beta \int \phi \, d\widehat{\mu}_{\mathcal{P}(\beta)} = \mathcal{P}(\beta) > 0.$$

This last inequality shows that δ_{0^∞} cannot be an equilibrium state, hence, there exists an equilibrium state which gives positive weight to $[1]$, and it is $\widehat{\mu}_{\mathcal{P}(\beta)}$.

For $\beta = \beta_c$, this depends on the value of α.

2.2.4. *Some more informations.* If the potential A is continuous, one can prove that $Z_c(\beta)$ is the pressure of the *dotted system with hole J*. Actually one consider the set Σ_J of point in Σ whose orbit never enter into J. Then, $Z_c(\beta)$ is the pressure for the system $(\Sigma_{J,\sigma})$ and the potential $\beta.A$.

e.g. For the Hofbauer potential and $J = [1]$, the dotted system in $\{0^\infty\}$.

It is thus easy to check that $\beta \mapsto Z_c(\beta)$ is a convex map. The pressure function $\beta \mapsto \mathcal{P}(\beta)$ is also convex. Due to Theorem 5 we emphasize the implicit function

$$\lambda_{\mathcal{P}(\beta),\beta} = 1.$$

3. Freezing phase transition with ground state supported into a quasi-crystal

3.1. Settings and results.

3.1.1. *Settings and one result.* Let $\mathcal{A}$ be a finite set with cardinality $D \geq 2$ called the alphabet. The associated shift $\mathcal{A}^{\mathbb{N}}$ will still be denoted by Σ. If $u = u_0 \dots u_{n-1}$ is a finite word and $v = v_0 \dots$ is a word, the concatenation uv is the new word $u_0 \dots u_{n-1} v_0 \dots$. If v is a finite word, v^n denotes the concatenated word

$$v^n = \underbrace{v \dots v}_{n \text{ times}}.$$

A substitution H is a map from an alphabet $\mathcal{A}$ to the set $\mathcal{A}^* \setminus \{\epsilon\}$ of nonempty finite words on $\mathcal{A}$. It extends to a morphism of $\mathcal{A}^*$ by concatenation, that is $H(uv) = H(u)H(v)$.

We refer to [18] for basic notion on substitutions.

Definition 3.1. *If H is a substitution, its* incidence matrix *is the $D \times D$ matrix $\mathcal{M}_H$ with entries a_{ij} where a_{ij} is the number of j's in $H(i)$. Then, H is said to be* primitive *if all entries of $\mathcal{M}_H^k$ are positive for some $k \geq 1$.*

A k-periodic point of H is an infinite word u with $H^k(u) = u$ for some $k > 0$. If $k = 1$ the point is said to be fixed. Then, H is said to be aperiodic *if no fixed point for H is a periodic sequence for σ.*

We emphasize an equivalent definition for being primitive. The substitution H is primitive if and only if there exists an integer k such that for every couple of letters (i, j), j appears in $H^k(i)$.

Let H be a substitution over the alphabet $\mathcal{A}$, and a be a letter such that $H(a)$ begins with a and $|H(a)| \geq 2$. Then there exists a unique fixed point u of H beginning with a (see [18, 1.2.6]). This infinite word is the limit of the sequence of finite words $H^n(a)$. Assume that ω is a fixed point for H, then we set

$$\mathbb{K} := \overline{\{\sigma^n(\omega),\ n \in \mathbb{N}\}}.$$

If H is a primitive substitution, then $\mathbb{K}$ does not depend on the the fixed point ω. If H is aperiodic, then $\mathbb{K}$ is uniquely ergodic but not reduced to a σ-periodic orbit. In that case, the unique σ-invariant probability is denoted by $\mu_{\mathbb{K}}$

We recall that the *language of a primitive substitution* is the set of finite words which appear in a fixed point. It is denoted by $\mathcal{L}_H$.

Definition 3.2. *A substitution is said to be* 2-full *if any word of length 2 in $\mathcal{A}^*$ belongs to the language of the substitution. A substitution is said to be* **marked** *if* *the set of the first (and last) letters of the images of the letters by the substitution is in bijection with the alphabet.*

We emphasize that the Thue-Morse substitution $H : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases}$ satisfies our assumptions. More precisely, the Thue Morse substitution and its language $\mathcal{L}$ fulfill:

- H is 2-full and marked.
- The non uniquely desubstituable words of $\mathcal{L}$ are $010, 101$.
- Every word of length at least 5 in $\mathcal{L}$ is uniquely desubstituable inside the language.
- The fixed point which begins by 0 can be written
$$u = 01.10.10.01.10.01.01.10.10.01.01\dots$$
Both fixed points are obtained by the repetition of the rule "block $\overline{\text{block}}$ $\overline{\text{block}}$ block".
- The language contains the words $\begin{cases} 0, 1 \\ 00, 01, 10, 11 \\ 001, 010, 011, 100, 101, 110 \end{cases}$

We refer to [18] and [5] for proofs.

The main result we want to present is the following:

Theorem 6 ([2])**.** *Consider a primitive 2-full aperiodic and marked substitution,* $\mathbb{K}$ *associated to* H *as above and consider a potential* $V := -\varphi$ *with* $\varphi(x) = \dfrac{1}{n} + o(\dfrac{1}{n})$ *if* $d(x, \mathbb{K}) = 2^{-n}$*. Then there exists a positive number* β_c *such that the pressure function has a freezing phase transition at* β_c*. More precisely:*

- *For* $\beta < \beta_c$ *the pressure function is analytic, there is a unique equilibrium state for* $\beta.V$ *and it has full support.*
- *For* $\beta > \beta_c$ *the pressure is equal to zero and* $\mu_{\mathbb{K}}$ *is the unique equilibrium state for* $\beta.V$*.*

3.1.2. *Some other results.* Similar results have been obtained for other substitutions. In [4] authors study the Fibonacci case (which is not marked). This has been extended in [10] to k-bonacci substitutions. In [**?**] the following meta-theorem is given:

Theorem 7. *Consider a shift* $\mathbb{K}$ *with finite alphabet which satisfies the following properties:*

(1) *It is linearly recurrent (see* [7, Sec7]*).*
(2) *The bispecial words are all of the length* $c.\lambda^n + o(\lambda^n)$*, where* $\lambda > 1$ *and* c *belongs to a finite set.*

(3) *Bispecial words cannot overlap each other for more than a fixed proportion than the smaller one*

Then, every non-negative potential of the form $\varphi(x) = 0$ *if and only if* x *belongs to* $\mathbb{K}$ *and* $\varphi(x) = -\frac{1}{n} + o(\frac{1}{n})$ *if* $d(x, \mathbb{K}) = 2^{-n}$ *admits a freezing phase transition with ground state supported into* $\mathbb{K}$.

3.2. Elements of the proof.

3.2.1. *Inducing operator and spectral radius.* We consider some word w_J which does not belong to the language of the substitution. In other words, the cylinder $[w_J]$ has empty intersection with $\mathbb{K}$.

Then we consider the potential $V(x) := -\log\left(\dfrac{n+1}{n}\right)$ if $d(x, \mathbb{K}) = 2^{-n}$ (with $n > 0$). Note that by assumption $d(x, \mathbb{K}) \leq 2^{-2}$ because H is 2-full.

We consider the induced transfer operator $\mathcal{K}_{Z,\beta}$ on $J := [w_J]$ associated to that potential V.

Lemma 3.3. *The spectral radius for* $\mathcal{L}_{Z,\beta}$ *is defined by* $\lambda_{Z,\beta} := \mathcal{L}_{Z,\beta}(\mathbb{1}_J)(x)$ *for every* x *in* J.

Proof. Note that by construction if x belongs to J and has first return time n, then for every $k \leq n-1$, $d(\sigma^k(x), \mathbb{K}) \geq 2^{-(n-k+|w_J|)}$. In other words, the maximal coincidence of any point $\sigma^k(x)$ with a word in the language $\mathcal{L}$ of the substitution is lower than $n - k + |w_J|$.

This means that if $y = wx$ is a J with first return time in J $|w|$, then for every x' in J, and for every $k \leq |w|$,

$$d(\sigma^k(wx'), \mathbb{K}) = d(\sigma^k(wx), \mathbb{K}).$$

Therefore for every x and x' in J,

$$\mathcal{L}_{Z,\beta}(\mathbb{1}_J)(x) = \mathcal{L}_{Z,\beta}(\mathbb{1}_J)(x').$$

From this equality, it follows that $\mathcal{L}_{Z,\beta}(\mathbb{1})$ is a constant function and then

$$\mathcal{L}^n_{Z,\beta}(\mathbb{1}) = (\mathcal{L}_{Z,\beta}(\mathbb{1}))^n$$

which shows that $\lambda_{Z,\beta}$ is equal to $\mathcal{L}_{Z,\beta}$. □

For each β there exists a critical $Z_c(\beta)$ such that $\mathcal{L}_{Z,\beta}$ is well defined for $Z > Z_c(\beta)$ and does not exist for $Z < Z_c(\beta)$.

We claim that $Z_c(\beta)$ is non-negative because it has to be larger or equal to $h_{\mu_{\mathbb{K}}} + \int V \, d\mu_{\mathbb{K}} = 0$. Our goal is to show that for β_c large enough, $\lambda_{0,\beta_c} < 1$. This will show that

- on the one hand $Z_c(\beta)=0$ for every $\beta\geq\beta_c$ (because it is non-negative and non-positive),
- no equilibrium state can give positive weight to J.

Lemma 3.4. *If no equilibrium state gives positive weight to J, then $\mu_{\mathbb{K}}$ is the unique equilibrium state.*

Proof. We do the proof by contradiction. Assume there is some equilibrium state different to $\mu_{\mathbb{K}}$. Pick some cylinder J' with empty intersection with $\mathbb{K}$ with positive weight (for this equilibrium state). Then we can induce on J' and we should be in case 1 or 2 of Theorem 5. Then, there exists an equilibrium state with full support and J has positive measure for this equilibrium state. □

3.2.2. *Excursion Free words.* Now let N be the integer such that $d(J,\mathbb{K})>D^{-N}$, and consider

$$R=-\log\left(1+\frac{1}{N}\right).$$

The integer N is a parameter that can be fixed as big as needed. For a fixed N, we define two classes of integers for each return word u: the **u-free** and the **u-excursions**. An integer $k\in[0,n-1]$ is u-free if $\delta(u_k\ldots u_{n-1}w_J)\leq N$. The integers between two consecutive u-free integers are called u-excursions. Remark that 0 is u-free by definition of N.

We fix $N>l(H)$, where $l(H)$ was defined in Theorem **??**. Then, every bispecial word that appear during an excursion has length bigger than $l(H)$.

Remark 6. *The terminology free and excursion words are used in order to have in mind some points far from $\mathbb{K}$ and some points close to $\mathbb{K}$. Actually, when points are far from $\mathbb{K}$ the digits may appear randomly as we are in the full shift $\mathcal{A}^{\mathbb{N}}$. On the contrary, when points are close to $\mathbb{K}$ the digits must obey for a while to the language $\mathcal{L}$.* ■

A word w is said to be **excursion free** if we can write $w=EF$ such that the integers inside $[0,|E|]$ are w-excursions and those inside $[|E|+1,|E|+|F|]$ are w-free. The set of all these words is denoted by $\mathcal{EF}$.

Let us denote the following quantity [1]

$$C_{\mathcal{EF}}=\sum_{w\in\mathcal{EF}}e^{-\beta S_{|w|}\varphi(w)}. \tag{17}$$

Proposition 3.5. *Let J be a cylinder outside $\mathbb{K}$ defined by the word w_J et $x\in J$. Assume that $C_{\mathcal{EF}}$ (cf Equation (17)) is finite, then we have*

$$\mathcal{L}_{0,\beta}(\mathbb{1}_J)(x)\leq\sum_{k\geq0}C^k_{\mathcal{EF}}\sum_{n\geq0}e^{(n+1)(\beta R+\log D)}.$$

[1] In all the following we make computations in $\overline{\mathbb{R}}$ since we have positive terms. It allows us to avoid problems of convergence of series.

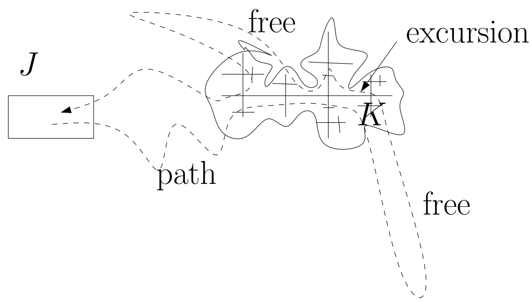

FIGURE 3. Path with free moments and excursions

Proof. Consider a path starting from J, free at the beginning and at the end. In between it alternates the words excursion-free, see Figure 3. Let us denote these excursions-free words by $E_iF_i, i \leq k$:

$$u = F_0(E_1F_1)(E_2F_2)\dots(E_kF_k).$$

Use the cocycle property for the Birkhoff sum. □

3.2.3. *Bispecial words and accidents.* Let x be an element of $\mathcal{A}^{\mathbb{N}}$ which does not belong to $\mathbb{K}$, then we define and denote:

- The word w is the maximal prefix of x such that w belongs to the language of $\mathbb{K}$. Thus we denote $d(x,\mathbb{K}) = D^{-d}, x = x_1\dots x_d\dots$ with $w = x_1\dots x_d$. Let us denote $\delta(x) = d$, and $\delta_k^n = \delta(\sigma^k \circ H^n(x))$ for every integers k and n. Note that $\delta = \delta_0^0$.
- If there exists an integer $b < d$ such that $\delta_b^0(x) > d - b$ and $\delta_i^0(x) = d - i$ for all $i < b$, then we say that an **accident appears at time** b. The prefix of $\sigma^b(x)$ of length δ_b^0 is called the first **accident** of x of **depth** δ_b^0.

Remark that the word w is non-empty since every letter is in the language of $\mathbb{K}$ if the substitution is primitive. Then, w is the unique word such that

$$x = wx', w \in \mathcal{L}_H, wx'_0 \notin \mathcal{L}_H.$$

Figure 4 illustrates the next lemma.

Lemma 3.6. *Let x be an infinite word not in $\mathbb{K}$. Assume that $\delta(x) = d$ and that the first accident appears at time $0 < b \leq d$ then the accident $x_b \dots x_{d-1}$ is a bispecial word of $\mathcal{L}_H$.*

Remark 7. *If $\mathcal{A}$ has cardinality two, then $x_0\dots x_{d-1}$ is not right-special. Moreover, and always if $\mathcal{A}$ has cardinality two, if $x = \sigma(z)$ and there is an accident at time 1 for z, then $x_0\dots x_{d-1}$ is not left-special.* ■

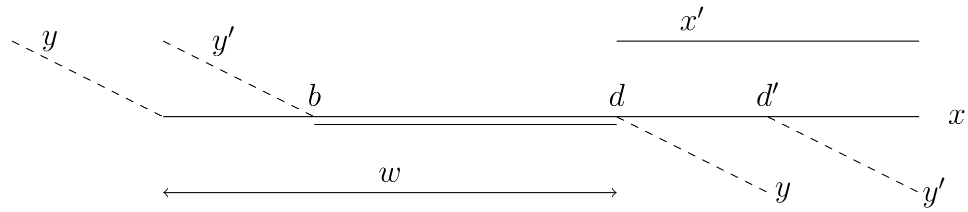

FIGURE 4. Accidents-dashed lines indicate infinite words in $\mathbb{K}$.

Definition 3.7. *We define inductively*

$$
\begin{aligned}
b_1 &= b = \min\{j \geq 1, d(\sigma^j x, \mathbb{K}) \leq d(\sigma^j x, \sigma^j(y))\} \\
b_2 &= \min\{j \geq 1, d(\sigma^{j+b_1} x, \mathbb{K}) \leq d(\sigma^{j+b_1} x, \sigma^j(y'))\} \\
b_3 &= \min\{j \geq 1, d(\sigma^{j+b_1+b_2} x, \mathbb{K}) \leq d(\sigma^{j+b_1+b_2} x, \sigma^j(y"))\} \\
&\dots
\end{aligned}
$$

We also define $B_j = b_0 + \dots b_j$ where $b_0 = 0$, and let us denote $d_j = \delta(\sigma^{B_{j-1}} x)$.

The integers $B_i, i \geq 1$ are called the times of accident, and the words $x_{b_i} \dots x_{b_i + b_{i+1}}$ are called **accidents**.

Proposition 3.8. *Let H be a primitive, aperiodic and marked substitution. There exists $l(H)$ such that if $\mathcal{W}_b$ denotes the set of bispecial words of length less than $l(H)$, then, every bispecial word can be written $H^n(v)$ with $v \in \mathcal{W}_b$ and n some integer.*

Remark 8. *We choose N to be large enough such that $N >> l(H)$ holds.*

We call λ the dominating eigenvalue for the incidence matrix of H. Then Proposition 3.8 yields:

Corollary 3.9. *There exist $0 < \theta < \lambda$ and a finite set of positive numbers c, such that the lengths of the bispecial words of $\mathcal{L}_H$ are of the form $c\lambda^n + O(\theta^n)$, $n \in \mathbb{N}$.*

Note that the numbers c are the lengths of the words in $\mathcal{W}_b$.

3.2.4. *main step: how to count excursions.* Note that $\displaystyle\sum_{n \geq 0} e^{(n+1)(\beta R + \log D)}$ can be as small as wanted if β increases because $\beta.R$ goes to $-\infty$. Then the goal is to prove that for any β big enough $C_{\mathcal{EF}}$ is smaller than 1.

$C_{\mathcal{EF}}$ is the contribution of infinitely many paths. Accidents are the source of increase for $C_{\mathcal{EF}}$: they may be very long paths with contribution $(S_n(V)(x))$ very close to 0.

We point out that in a Excursion-Free path, the Free part is easy to bound as we did just above.

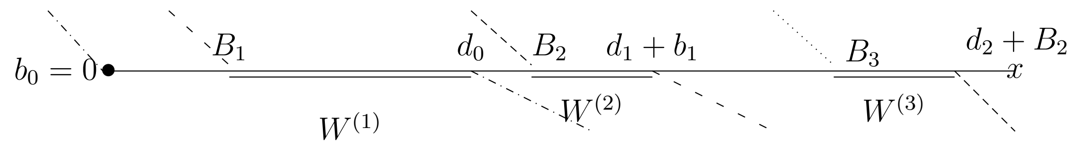

Figure 5. Accidents inside an excursion-free word.

The main idea is to rank well all the Excursion paths. Instead of ranking them with respect to their respective length we will rank them with respect to the accidents:

(1) We first rank all the excursion-words with respect to the number M of accident(s) that they have. We call $C_{\mathcal{EF}}(M)$ the contribution of Excursion-Free paths with exactly M accidents. Note that $M \geq 1$ because the entrance in the excursion zone is only realized by an accident.
(2) Then, we rank the Excursion paths with M accident with respect of the bispecial words involved. Recall Proposition 3.8, therefore all the bispecial words involved have lenght $> l(H)$ and are thus of the form $H^k(v)$ with $v \in \mathcal{W}_b$.
(3) If we know the bispecial words W^i $i = 1, \dots M$ involved the path (excursion part) is of the form $W^1 T_1 W^2 T^2 \dots W^M \dots$. Then, the linear recurrence of $\mathbb{K}$ shows that there are a countable set of possible T^i's
$$T^i(0), T^i(1), \dots$$
with increasing sizes of the form
$$|T^i(j)| \sim |T^i(0)| + j|W^{i+1}|.$$

Then, the contribution of the path between two B_i and B_{i+1} is of the form
$$\left(\frac{d_i + 1 - b_{i+1}}{d_i + 1}\right)^\beta.$$

Ranking the path as said above one manage to show that $C_{\mathcal{EF}}(M)$ is bounded by a quantity r^M with $0 < r < 1$ for β large enough. Actually, one may show that $r \to 0$ if $\beta \to +\infty$.

References

[1] Alexandre Baraviera, Renaud Leplaideur, and Artur O. Lopes. The potential point of view for renormalization. *Stoch. Dyn.*, 12(4):1250005, 34, 2012.
[2] N. Bedaride, P. Hubert, and R. Leplaideur. Thermodynamic formalism and Substitutions. *ArXiv e-prints*, November 2015.
[3] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Lecture Notes in Mathematics, Vol. 470. Springer-Verlag, Berlin, 1975. 2nd ed. - 2008 by JR Chazottes.
[4] H. Bruin and R. Leplaideur. Renormalization, freezing phase transition and fibonacci quasicrystals. *Annales de l'Ecole normale supérieure*, To appear, 2015.
[5] J. Cassaigne. Complexité et facteurs spéciaux. *Bull. Belg. Math. Soc. Simon Stevin*, 4(1):67–88, 1997. Journées Montoises (Mons, 1994).

[6] Yael Naim Dowker. Finite and σ-finite invariant measures. *Ann. of Math. (2)*, 54:595–608, 1951.

[7] F. Durand, B. Host, and C. Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergodic Theory Dynam. Systems*, 19(4):953–993, 1999.

[8] Richard S. Ellis. *Entropy, large deviations, and statistical mechanics.* Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1985 original.

[9] Richard S. Ellis and Kongming Wang. Limit theorems for the empirical vector of the Curie-Weiss-Potts model. *Stochastic Processes Appl.*, 35(1):59–79, 1990.

[10] J. Emme. Thermodynamic formalism and k-bonacci substitutions. *ArXiv e-prints*, February 2016.

[11] R. B. Griffiths and D. Ruelle. Strict convexity (continuity) of the pressure in lattice systems. *Commun. Math. Phys.*, 23:169–175, 1971.

[12] J. Hubbard. Calculation of partition functions. *Phys. Rev. Lett.*, 3:77–78, Jul 1959.

[13] Gerhard Keller. *Equilibrium states in ergodic theory.* Cambridge: Cambridge University Press, 1998.

[14] R. Leplaideur. From local to global equilibrium states: thermodynamic formalism via an inducing scheme. *Electron. Res. Announc. Math. Sci.*, 21:72–79, 2014.

[15] R. Leplaideur. Chaos : Butterflies also generate phase transitions and parallel universes. 2015.

[16] Steven Orey. Large deviations for the empirical field of Curie-Weiss models. *Stochastics*, 25(1):3–14, 1988.

[17] Karl Petersen. *Ergodic theory.* Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge, New York, 1983. Autre(s) tirage(s) : 1989 (avec corrections), 1991.

[18] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[19] D. Ruelle. *Thermodynamic formalism.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.

[20] R. L. Stratonovič. A method for the computation of quantum distribution functions. *Dokl. Akad. Nauk SSSR (N.S.)*, 115:1097–1100, 1957.

[21] M. Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.*, 37(4):303–314, 1980.

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