Freezing Phase Transition with ground state into a quasi-crystal

Renaud Leplaideur

UBO

date=today
In dynamical system a freezing transition phase occurs if the shape of the pressure function is of the form:

\[ P(\beta V) \]

Our goal is to show how we can construct such transitions phase where after the transition the unique equilibrium state is supported into a quasi-crystal (question due to van Enter).
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Our goal is to show how we can construct such transitions phase where after the transition the unique equilibrium state is supported into a quasi-crystal (question due to van Enter).
Aim: to do a course which could interest physicists.

Different settings:

Dynamical system consider a time-action: \( X \) compact metric space, \( T: X \to X \) continuous. Goal = describe orbits \( x, T(x), T(T(x)) = T^2(x), \ldots \).

The quasi-crystal involved are included into a one-dimensional lattice as e.g. \( \{0,1\}^\mathbb{Z} \) or \( \{0,1\}^\mathbb{N} \).

They will come from a substitution (e.g. Thue-Morse or the Fibonacci substitutions).

Physicists some times consider 1-d lattices as uninteresting.
However, the Curie-Weiss model naturally leads to 1-d lattice. Instead of considering interaction with the closest neighbors one consider that every atom interact with every other one. This can make sense (NPN or PNP). For this model, settings in statistical mechanics and in ergodic theory are very close.
Another problem: share the same vocabulary

1. Gibbs measures,
2. phase transition,
3. pressure,
4. equilibrium states,
5. ...

but it is not clear that all these notions coincide in both areas.

Necessary to make precise similitudes and differences, especially when settings are close.
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**Renaud (UBO)**

Renormalization
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   ▶ Link between 2 notions of Gibbs measures
   ▶ Link between phase transition in SM and # of ergodic components.

2 Basic notions & tools on Ergodic Theory
   ▶ Transfer Operator
   ▶ Inducing scheme (a way to detect freezing phase transitions).

Main issue : show contribution of paths.

3 A machinery to produce freezing phase transitions with ground state into a quasi-crystal. How to compute the contributions of paths.
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Open question

How to extend the machinery to higher-dimensional case ($\mathbb{Z}^d$-action)?
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Steven Orey.
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D. Ruelle.
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The mathematical structures of equilibrium statistical mechanics.

Y. Sinaǐ.
Curie-Weiss model with respect to Ergodic theory
We consider the sets $\Lambda = \{-1, +1\}$ and $\Sigma := \Lambda^\mathbb{N}$.
A point $x = x_0, x_1, \ldots \in \Sigma$ = infinite word where the $x_i$ are in $\Lambda$.
Notation $x = x_0x_1x_2\ldots$.

$x_i \in \Lambda$ can either be called a letter, or a digit or a symbol.
A point $w \in \Lambda^n$ is called a word of length $|w| = n$.
The concatenation of two words $w$ and $w'$ with $|w| < +\infty$ is the word $ww' = w_0 \ldots w_{n-1}w'_0w'_1\ldots$. 

If $\omega_0 \ldots \omega_{n-1}$ is a finite word, we set

$$H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} \omega_j \omega_i. \quad (1)$$

It is called the **Curie-Weiss Hamiltonian**.

The empirical magnetization for $\omega$ is $m_n(\omega) := \frac{1}{n} \sum_{j=0}^{n-1} \omega_j$.

$$H_n(\omega) = -\frac{n}{2} (m_n(\omega))^2.$$
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Set

- $\rho$ is the uniform measure on $\{-1, 1\}$, i.e. $\rho(\{1\}) = \rho(\{-1\}) = \frac{1}{2}$,
- $\mathbb{P} := \rho^{\otimes N}$ the product measure on $\Sigma$

**Definition**

**probabilistic Gibbs measure** (PGM for short) $\mu_{n, \beta}$ on $\Lambda^n$,

$$\mu_{n, \beta}(d\omega) := \frac{e^{-\beta \cdot H_n(\omega)}}{Z_{n, \beta}} \mathbb{P}(d\omega), \quad (2)$$

where $Z_{n, \beta}$ is the normalization factor

$$Z_{n, \beta} = \frac{1}{2^n} \sum_{\omega', \; |\omega'| = n} e^{-\beta \cdot H_n(\omega')}.$$
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\[
\sigma(x_0 x_1 x_2 \ldots) = x_1 x_2 \ldots
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**Definition**

A measure \( \tilde{\mu} \) on \( \Lambda^N \) is said to be \( \sigma \)-invariant if for every Borel set \( B \),

\[
\tilde{\mu}(\sigma^{-1}(B)) = \tilde{\mu}(B).
\]

A cylinder (of length \( n \)) is denoted by \([x_0 \ldots x_{n-1}]\). It is the set of points \( y \) such that \( y_i = x_i \) for \( i = 0, \ldots, n - 1 \). A finite word \( x = x_0 \ldots x_{n-1} \) generates a cylinder \([x]\).

**Claim**

Cylinders generate Borel \( \sigma \)-algebra.

Therefore \( \tilde{\mu} \) is \( \sigma \)-invariant if and only if for every finite word \( x \),

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Definition

Let \( \phi : \Sigma \to \mathbb{R} \) be Lipschitz continuous \( (\star) \). An invariant measure \( \tilde{\mu} \) is said to be a dynamical Gibbs measure \( \text{(DGM for short)} \) if there exist \( C = C(\phi) > 0 \) and \( P \) such that for \( x \in \Sigma \) and every \( n \),

\[
e^{-C} \leq \frac{\tilde{\mu}([x_0 \ldots x_{n-1}])}{e^{S_n(\phi)(x) -nP}} \leq e^C,
\]

(3)

where \( S_n(\phi)(x) \) stands for \( \phi(x) + \phi \circ \sigma(x) + \ldots + \phi \circ \sigma^{n-1}(x) \).

\( \star \)
Definition
For $\phi : \Sigma \to \mathbb{R}$ continuous and $\beta > 0$, the pressure function is defined by

$$P(\beta \cdot \phi) := \sup_{\mu} \left\{ h_\mu + \beta \int \phi \, d\mu \right\},$$

(4)

A measure which realizes the maximum is called an equilibrium state for $\phi$.

Theorem (Ruelle-Griffiths)
If $\phi$ is Lipschitz continuous then

1. there exists a unique equilibrium state for $\beta \cdot \phi$,
2. It is Dynamical Gibbs measure,
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Theorem (Orey, LW)

Let $\xi_\beta$ be the unique point in $[0, 1]$ which realizes the maximum for $\varphi_1(x) := \log(\cosh(\beta x)) - \frac{\beta}{2} x^2$. Let $\tilde{\mu}_b^+$ and $\tilde{\mu}_b^-$ be the dynamical Gibbs measures for $b.1_{[+1]}$ and $b.1_{[-1]}$ respectively. Then

$$
\mu_{n,\beta} \xrightarrow{w} n \to +\infty \begin{cases} 
\tilde{\mu}_0 & \text{if } \beta \leq 1, \\
\frac{1}{2} \left[ \tilde{\mu}_{2\beta,\xi_\beta}^+ + \tilde{\mu}_{2\beta,\xi_\beta}^- \right] & \text{if } \beta > 1.
\end{cases}
$$

(5)

Actually $\mu_{n,\beta} \to \frac{1}{2} \left[ \tilde{\mu}_{2\beta,\xi_\beta}^+ + \tilde{\mu}_{2\beta,\xi_\beta}^- \right]$ for every $\beta > 0$ since $\xi_\beta \equiv 0$ for $\beta \leq 1$.

It is said that there is a phase transition at $\beta = 1$. 

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\frac{1}{2} \left[ \tilde{\mu}_{2\beta}^+ \xi_\beta + \tilde{\mu}_{2\beta}^- \xi_\beta \right] & \text{if } \beta > 1.
\end{cases}
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(5)

Actually $\mu_{n,\beta} \rightarrow \frac{1}{2} \left[ \tilde{\mu}_{2\beta}^+ \xi_\beta + \tilde{\mu}_{2\beta}^- \xi_\beta \right]$ for every $\beta > 0$ since $\xi_\beta \equiv 0$ for $\beta \leq 1$.

It is said that there is a phase transition at $\beta = 1$. 

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**Theorem (Orey, LW)**

Let \( \xi_\beta \) be the unique point in \([0, 1]\) which realizes the maximum for \( \varphi_1(x) := \log(\cosh(\beta \cdot x)) - \frac{\beta}{2} x^2 \). Let \( \tilde{\mu}_b^+ \) and \( \tilde{\mu}_b^- \) be the dynamical Gibbs measures for \( b.1_{[+1]} \) and \( b.1_{[-1]} \) respectively. Then

\[
\mu_{n, \beta} \xrightarrow{w} n \to +\infty \begin{cases} 
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Actually \( \mu_{n, \beta} \to \frac{1}{2} \left[ \tilde{\mu}_2^+ \cdot \xi_\beta + \tilde{\mu}_2^- \cdot \xi_\beta \right] \) for every \( \beta > 0 \) since \( \xi_\beta \equiv 0 \) for \( \beta \leq 1 \).

It is said that there is a phase transition at \( \beta = 1 \).
To prove $\mu_{n,\beta} \rightarrow_{n \rightarrow +\infty} \tilde{\mu}$ it is sufficient to prove 
$\lim_{n \rightarrow +\infty} \mu_{n,\beta}([\omega]) = \tilde{\mu}([\omega])$ for any cylinder $\omega$. 

We will prove:

$$
\lim_{n \rightarrow \infty} \mu_{n,\beta}([\omega_0 \ldots \omega_{p-1}]) = \begin{cases} 
\frac{1}{2^p} & \text{if } \beta \leq 1, \\
1 & \frac{1}{2} (f(\xi_\beta) + f(-\xi_\beta)) & \text{if } \beta > 1,
\end{cases}
$$

(6)

where

$$
f(y) = \frac{e^{\beta y} S_p(\omega)}{(e^{\beta y} + e^{-\beta y})^p}.
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A more general result: Curie Weiss Potts model

Holds for $\Lambda = \{\theta^1, \ldots, \theta^q\}$ with $q > 2$.

$$H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} 1_{\omega_j = \omega_i}. \quad (7)$$

Set $L_n(\omega) = (L_{n,1}(\omega), \cdots, L_{n,q}(\omega))$ where

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Define the Probabilistic Gibbs measure $\mu_{n,\beta}$ on $\Sigma_q$ by

$$\mu_{n,\beta}(d\omega) := \frac{e^{-\beta \cdot H_n(\omega)}}{Z_{n,\beta}} \mathbb{P}(d\omega) = \frac{e^{\beta \|L_n(\omega)\|^2}}{Z_{n,\beta}} \mathbb{P}(d\omega), \quad (8)$$

where $Z_{n,\beta}$ is the normalization factor.
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Theorem (Weak convergence for the CWP model)

For $1 \leq k \leq q$, $b \in \mathbb{R}$, let $\tilde{\mu}_b^k$ be the dynamical Gibbs measure for $b \mathbb{1}_{[\theta^k]}$. Let $\beta_c = \frac{2(q-1) \log(q-1)}{q-2}$. For $0 < \beta < \beta_c$ set $s_\beta = 0$ and for $\beta \geq \beta_c$ let $s_\beta$ be the largest solution of the equation

$$s = \frac{e^{\beta s} - 1}{e^{\beta s} + q - 1}.$$ \hspace{1cm} (9)

Then,

$$\rho \otimes N = \tilde{\mu}_0 \quad \text{if } 0 < \beta < \beta_c,$$

$$\frac{1}{q} \sum_{k=1}^{q} \tilde{\mu}_b^k \quad \text{if } \beta = \beta_c,$$

$$\frac{A \tilde{\mu}_0 + B \sum_{k=1}^{q} \tilde{\mu}_b^k \cdot s_\beta}{A + qB} \quad \text{if } \beta = \beta_c.$$ \hspace{1cm} (10)
Construction of dynamical (local) equilibrium states
Recall : Gibbs measure $\forall x \in \Sigma$, $\forall n$,

$$e^{-c} \leq \frac{\tilde{\mu}([x_0 \ldots x_{n-1}])}{e^{S_n(\phi)(x)-nP}} \leq e^{c}.$$ 

Pressure + equilibrium state :

$$\mathcal{P}(\beta, \phi) := \sup_{\mu} \left\{ h_{\mu} + \beta \int \phi \, d\mu \right\}.$$ 

Ruelle-Griffiths theorem : If $\phi$ is Lipschitz continuous unique equilibrium state= Gibbs state + $P(\beta)$ analytic.

$\implies$ to get equilibrium state one constructs Gibbs measures!
Recall : Gibbs measure $\forall x \in \Sigma, \forall n,$

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Definition

Transfer operator for potential $\mathcal{A}$,

$$
\mathcal{L}_\mathcal{A}(\varphi)(x_0x_1\cdots) = \sum_{y,\sigma(y)=x} e^{\mathcal{A}(y)} \varphi(y)
= \sum_{a,a_0 \in \mathcal{A}} e^{\mathcal{A}(ax_0x_1x_2,\cdots)} \varphi(ax_0x_1x_2\cdots).
$$

Also called Ruelle-Perron-Frobenius operator.

1. $\mathcal{L}_\mathcal{A}$ acts on continuous and Lipschitz functions.
2. Spectral radius $\lambda_\mathcal{A}$ is an eigenvalue for $\mathcal{L}_\mathcal{A}$ and $\mathcal{L}_\mathcal{A}^*$.
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   - $\mu_\mathcal{A} := H_\mathcal{A} \nu_\mathcal{A}$ is Gibbs measure.
   - $\lambda_\mathcal{A} = e^{P(A)}$.

Spectral Gap $\Rightarrow$ Analyticity.
Definition

Transfer operator for potential $A$,

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\mathcal{L}_A(\varphi)(x_0x_1\cdots) = \sum_{y, \sigma(y) = x} e^A(y) \varphi(y) = \sum_{a, ax_0 \in A} e^A(ax_0x_1x_2\cdots) \varphi(ax_0x_1x_2\cdots).
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Spectral Gap $\implies$ Analyticity.
If $A$ only depends on 2 coordinates $A(x_0x_1x_2\ldots) = A(x_0, x_1)$, then $\mathcal{L}_A$ is a $2 \times 2$ matrix.

**Theorem**

*(Perron-Frobenius)* Let $B = (b_{ij})$ be a $d \times d$ matrix with positive entries. Then, the spectral radius of $B$, say $\lambda$, is a simple dominated eigenvalue. The associated eigenspace is generated by some “positive” vector $u = (u_1, \cdots, u_d)$ with $u_i > 0$. 
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Let $M = (M_{ij})$ be the matrix with entries $e^{A(i,j)}$. Let $r = (r_1, \ldots, r_d)$ be the right-eigenvector associated to $\lambda$ with normalization $\sum r_i = 1$. Let $l = (l_1, \ldots, l_d)$ be the left-eigenvector for $\lambda$ with renormalization $\sum l_i r_i = 1$. Then, $r$ is the eigenmeasure $\nu_A$ and $l$ is the density $H_A$.

The Gibbs measure of the cylinder $[i_0 \ldots i_{n-1}]$ is $\mu_A = ([i_0 \ldots i_{n-1}]) = l_{i_0} e^{S_n(A)(x) - n \log \lambda_A r_{i_{n-1}}}$. 
(freezing) phase transition?

**Definition**

We say that there is a transition phase at $\beta_0$ if the pressure function $P(\beta)$ is not analytic at $\beta_0$. We say that it is a *freezing phase transition* if $P(\beta)$ is affine for every $\beta > \beta_c$.

In case of a freezing phase transition, after transition

$$P(\beta) = h + \beta a,$$

with $a = \max \int A \, d\mu$.

In ergodic theory, a measure satisfying this last property is said to be *A-maximizing*. In Statistical Mechanics it is a *ground state*. $h$ = residual entropy = max of $h_\mu$, $\mu$ is A-maximizing.
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Remarque

For Curie Weiss model phase transition means more than one “ergodic” components for $\lim_{n \to +\infty} \mu_{n,\beta}$. Here, phase transition $\perp \#$ of equilibrium states.

Examples with $\mathcal{P}(\beta)$ analytic but several equilibrium and examples with $\mathcal{P}(\beta)$ non-analytic but a unique equilibrium.

Question

Can get freezing phase transition with support in a quasi-crystal set after the transition?
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For Curie Weiss model phase transition means more than one “ergodic” components for $\lim_{n \to +\infty} \mu_{n,\beta}$. Here, phase transition $\perp \#$ of equilibrium states.

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How to detect/get freezing phase transition?

Via an inducing scheme. We consider a cylinder $J = [w, J]$ in $\Sigma$. For $x$ in $J$,

- the first return time is $\tau(x) = \min\{n \geq 1, \sigma^n(x) \in J\} \leq +\infty$.
- First return map $F(x) := \sigma^{\tau(x)}(x)$. This map is well defined if $\tau(x) < +\infty$.
- The main important point is that the inverse branches are well defined everywhere on $J$.

**Theorem (Dowker)**

If $\hat{\mu}$ is a $\sigma$-invariant probability measure with $\hat{\mu}(J) > 0$, then, the conditional measure $\mu := \frac{\hat{\mu}(\cdot \cap J)}{\hat{\mu}(J)}$ is $F$-invariant.

Conversely, if $\mu$ is a $F$-invariant probability measure and $\int \tau \, d\mu < +\infty$, then there exists a unique $\sigma$-invariant probability $\hat{\mu}$ such that $\mu := \frac{\hat{\mu}(\cdot \cap J)}{\hat{\mu}(J)}$ holds.
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At that stage we have two different dynamical systems: \((\Sigma, \sigma)\) and \((J, F)\).

By the Abramov formula

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\hat{h}_\mu(f) + \int A \, d\hat{\mu} \leq \mathcal{P} \quad \text{with equality iff } \hat{\mu} = \text{equil. state}
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\hat{h}_\mu(f) + \int A \, d\hat{\mu} - \mathcal{P} \leq 0 \quad \text{with equality iff } \hat{\mu} = \text{equil. state}
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Consequence

The thermodynamic formalism for \((\Sigma, \sigma)\) and potential \(A\) is related to the thermodynamic formalism for \((J, F)\) and \(S_{\tau(.)}(A)(.)\).
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The thermodynamic formalism for $(\Sigma, \sigma)$ and potential $A$ is related to the thermodynamic formalism for $(J, F)$ and $S_{\tau(.)}(A)(.)$. 
Let $A : \Sigma \to \mathbb{R}$ be some potential function and $J = [w_J]$ be a cylinder. Consider the first return map to $J$, with return time
\[ \tau(x) = \min \{ n \geq 1, \sigma^n(x) \in J \}. \]
Then we define, for each $\beta > 0$ and $Z \in \mathbb{R}$, an induced transfer operator by:
\[
\mathcal{L}_{Z, \beta}(g)(x) = \sum_{n \in \mathbb{N}} \sum_{\substack{\tau(y) = n \\ \sigma^n(y) = x}} e^{\beta \cdot S_n(A)(y) - nZ} g(y)
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where $S_N(A)(y) = \sum_{k=0}^{N-1} A \circ \sigma^k(y)$ and $g$ is a continuous function from $J$ to $\mathbb{R}$.
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where $S_N(A)(y) = \sum_{k=0}^{N-1} A \circ \sigma^k(y)$ and $g$ is a continuous function from $J$ to $\mathbb{R}$.
Theorem

We have with the previous notations:

- For every $\beta \geq 0$, there exists a minimal $Z_c(\beta) \in \mathbb{R} \cup \{-\infty\}$ such that for every $Z > Z_c(\beta)$, $L_{Z,\beta}$ acts on $C^0(J)$. In particular, for every $Z > Z_c(\beta)$, for every $x \in J$ and for every $g \in C^0(J)$, $L_{Z,\beta}(g)(x)$ converges.
- $P(\beta) \geq Z_c(\beta)$.
- Let $\lambda_{Z,\beta}$ be the spectral radius for $L_{Z,\beta}$ and for $Z > Z_c(\beta)$. Then $Z \mapsto \log \lambda_{Z,\beta}$ is a decreasing function and we have three possible cases given by

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\begin{align*}
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Theorem (suite)

- If case 1 holds, then $\log \lambda_{Z,\beta} = 0$ if and only if $Z = \mathcal{P}(\beta)$ and there is a unique equilibrium state for $\beta$. It is a fully supported measure in $\Sigma$. Moreover, $Z_c(\beta) < \mathcal{P}(\beta)$ and $\beta \to \mathcal{P}(\beta)$ is analytic on the largest open interval where case 1 holds.

- If case 3 holds, then no equilibrium state gives positive weight to $J$.

- Case 2 is the critical case.

Example: Hofbauer.
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My lectures in cartoons
Ergodic Gibbs Measure $\approx$ Limit of Gibbs measures for Curie-Weiss Model
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Critical level
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measure contribution
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measure contribution
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Measure contribution

Measure contribution

Measure contribution

Measure sum of contributions

Measure contribution
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Z & < Z_c(\beta)
\end{align*}
\]
\[ \lambda_{Z,\beta} \approx \mathcal{L}_{Z,\beta}(1)(x) = \sum_{n \in \mathbb{N}} \sum_{\tau(y) = n} e^{\beta \cdot S_n(A)(y) - nZ} \]
Play on the temperature $\beta$

$\approx \Phi$ (potential)

Theorem

There exist sophisticated baskets such that if $\beta$ is sufficiently big (temperature sufficiently small) the bees do no collect too much contaminated pollen.
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Theorem

Consider a primitive, aperiodic and marked substitution, $\mathbb{K}$ the attractor associated to $H$ and consider the potential defined by

$$V(x) := -\log \left( \frac{n + 1}{n} \right) \text{ if } d(x, \mathbb{K}) = 2^{-n}.$$ Then there exists a positive number $\beta_c$ such that the pressure function has a freezing phase transition at $\beta_c$. More precisely:

- For $\beta < \beta_c$ the pressure function is analytic, there is a unique equilibrium state for $\beta.V$ and it has full support.
- For $\beta > \beta_c$ the pressure is equal to zero and $\mu_{\mathbb{K}}$ is the unique equilibrium state for $\beta.V$. 