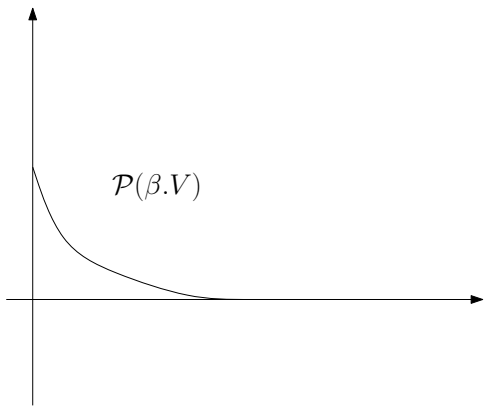
Freezing Phase Transition with ground state into a quasi-crystal

Renaud Leplaideur

UBO

date=today

In dynamical system a freezing transition phase occurs if the shape of the pressure function is of the form :



Our goal is to show how we can construct such transitions phase where after the transition the unique equilibrium state is supported into a quasi-crystal (question due to van Enter).

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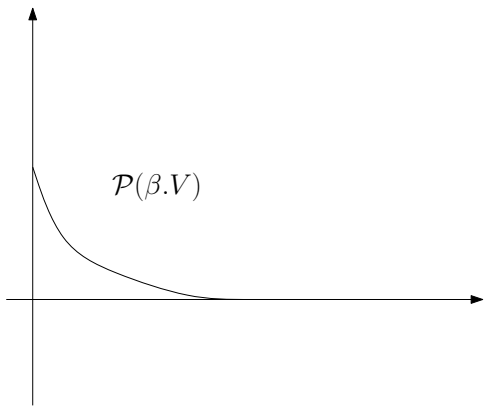

Our goal is to show how we can construct such transitions phase where after the transition the unique equilibrium state is supported into a quasi-crystal (question due to van Enter).

Aim : to do a course which could interest physicists.
Different settings :
Dynamical system consider a time-action : X compact metric space, $T : X \to X$ continuous. Goal = describe orbits $x, T(x), T(T(x)) = T^2(x), \ldots$.
The quasi-crystal involved are included into a one-dimensional lattice as *e.g.* $\{0,1\}^{\mathbb{Z}}$ or $\{0,1\}^{\mathbb{N}}$.
They will come from a substitution (*e.g.* Thue-Morse or the Fibonacci substitutions).
Physicists some times consider 1-d lattices as uninteresting.

However, the Curie-Weiss model naturally leads to 1-d lattice.
Instead of considering interaction with the closest neighbors one consider that every atom interact with every other one.
This can make sense (NPN or PNP).
For this model, settings in statistical mechanics and in ergodic theory are very close.

Another problem : share the same vocabulary

1. Gibbs measures,
2. phase transition,
3. pressure,
4. equilibrium states,
5. ...

but it is not clear that all these notions coincide in both areas.

Necessary to make precise similitudes and differences, especially when settings are close.

Stat. Mecha.	Erg. Theo.
Sites , finite box Λ^n	infinite lattice $\Lambda^{\mathbb{N}}$
Hamiltonian (Interaction between sites)	Potential (function $\Lambda^{\mathbb{N}} \to \mathbb{R}$)
Gibbs measure on Λ^n	Gibbs measure on $\Lambda^{\mathbb{N}}$
Phase transition $\approx$ discontinuity on $\#\lim_{n\to+\infty}$ Gibbs measure	Lack of analyticity of the pressure function

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Plan of the course

1. Curie Weiss model vs Ergodic theory
   - Link between 2 notions of Gibbs measures
   - Link between phase transition in SM and # of ergodic components.
2. Basic notions & tools on Ergodic Theory
   - Transfer Operator
   - Inducing scheme (a way to detect freezing phase transitions).

   Main issue : show contribution of paths.
3. A machinery to produce freezing phase transitions with ground state into a quasi-crystal. How to compute the contributions of paths.

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Open question

How to extend the machinery to higher-dimensional case ($\mathbb{Z}^d$-action) ?

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Curie-Weiss model with respect to Ergodic theory

We consider the sets $\Lambda = \{-1, +1\}$ and $\Sigma := \Lambda^{\mathbb{N}}$.
A point $x = x_0, x_1, \ldots \in \Sigma$ =infinite word where the x_i are in Λ.
Notation $x = x_0 x_1 x_2 \ldots$.
$x_i \in \Lambda$ can either be called a letter, or a digit or a symbol.
A point $w \in \Lambda^n$ is called a word of length $|w| = n$.
The concatenation of two words w and w' with $|w| < +\infty$ is the word $ww' = w_0 \ldots w_{n-1} w'_0 w'_1 \ldots$.

If $\omega_0 \dots \omega_{n-1}$ is a finite word, we set

$$H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} \omega_j \omega_i. \tag{1}$$

It is called the Curie-Weiss Hamiltonian.

The empirical magnetization for ω is $m_n(\omega) := \frac{1}{n} \sum_{j=0}^{n-1} \omega_j$.

$$H_n(\omega) = -\frac{n}{2}(m_n(\omega))^2.$$

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Set

- ρ is the uniform measure on $\{-1,1\}$, i.e. $\rho(\{1\}) = \rho(\{-1\}) = \frac{1}{2}$,
- $\mathbb{P} := \rho^{\otimes \mathbb{N}}$ the product measure on Σ

Definition

probabilistic Gibbs measure (PGM for short) $\mu_{n,\beta}$ on Λ^n,

$$\mu_{n,\beta}(d\omega) := \frac{e^{-\beta.H_n(\omega)}}{Z_{n,\beta}} \mathbb{P}(d\omega), \tag{2}$$

where $Z_{n,\beta}$ is the normalization factor

$$Z_{n,\beta} = \frac{1}{2^n} \sum_{\omega',\ |\omega'|=n} e^{-\beta.H_n(\omega')}.$$

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The shift map σ is defined on Σ by

$$\sigma(x_0 x_1 x_2 \ldots) = x_1 x_2 \ldots.$$

Definition

A measure $\widetilde{\mu}$ on $\Lambda^{\mathbb{N}}$ is said to be σ-invariant if for every Borel set B,

$$\widetilde{\mu}(\sigma^{-1}(B)) = \widetilde{\mu}(B).$$

A *cylinder* (of length n) is denoted by $[x_0 \ldots x_{n-1}]$. It is the set of points y such that $y_i = x_i$ for $i = 0, \ldots n-1$. A finite word $x = x_0 \ldots x_{n-1}$ generates a cylinder $[x]$.

Claim

Cylinders generate Borel σ-algebra.

Therefore $\widetilde{\mu}$ is σ-invariant if and only if for every finite word x,

$$\widetilde{\mu}([0x]) + \widetilde{\mu}([1x]) = \widetilde{\mu}([x]).$$

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Definition

Let $\phi : \Sigma \to \mathbb{R}$ be Lipschitz continuous ($\star$). An invariant measure $\widetilde{\mu}$ is said to be a *dynamical Gibbs measure* (DGM for short) if there exist $C = C(\phi) > 0$ and $\mathcal{P}$ such that for $x \in \Sigma$ and every n,

$$e^{-C} \leq \frac{\widetilde{\mu}([x_0 \ldots x_{n-1}])}{e^{S_n(\phi)(x) - n\mathcal{P}}} \leq e^{C}, \tag{3}$$

where $S_n(\phi)(x)$ stands for $\phi(x) + \phi \circ \sigma(x) + \ldots + \phi \circ \sigma^{n-1}(x)$.

$\star$

Definition

For $\phi : \Sigma \to \mathbb{R}$ continuous and $\beta > 0$, the *pressure function* is defined by

$$\mathcal{P}(\beta.\phi) := \sup_{\mu} \left\{ h_\mu + \beta \int \phi \, d\mu \right\}, \tag{4}$$

A measure which realizes the maximum is called an equilibrium state for ϕ.

Theorem (Ruelle-Griffiths)

If ϕ is Lipschitz continuous then

1. *there exists a unique equilibrium state for $\beta.\phi$,*
2. *It is Dynamical Gibbs measure,*
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Let ξ_β be the unique point in $[0,1]$ which realizes the maximum for $\varphi_I(x) := \log(\cosh(\beta.x)) - \frac{\beta}{2}x^2$. Let $\widetilde{\mu}_b^+$ and $\widetilde{\mu}_b^-$ be the dynamical Gibbs measures for $b.\mathbb{1}_{[+1]}$ and $b.\mathbb{1}_{[-1]}$ respectively. Then

$$\mu_{n,\beta} \xrightarrow[n\to+\infty]{w} \begin{cases} \widetilde{\mu}_0 & \text{if } \beta \leq 1, \\ \frac{1}{2}\left[\widetilde{\mu}^+_{2\beta.\xi_\beta} + \widetilde{\mu}^-_{2\beta.\xi_\beta}\right] & \text{if } \beta > 1. \end{cases} \tag{5}$$

Actually $\mu_{n,\beta} \to \frac{1}{2}\left[\widetilde{\mu}^+_{2\beta.\xi_\beta} + \widetilde{\mu}^-_{2\beta.\xi_\beta}\right]$ for every $\beta > 0$ since $\xi_\beta \equiv 0$ for $\beta \leq 1$.

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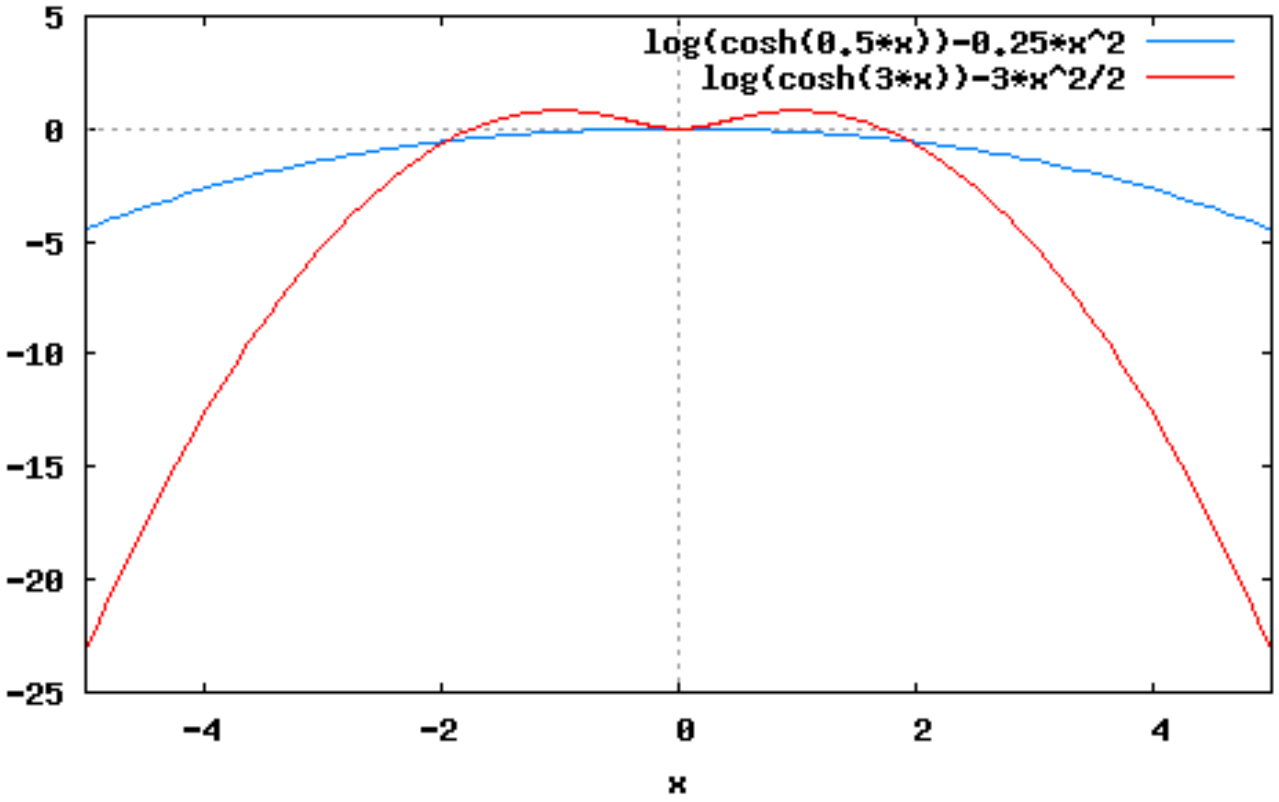
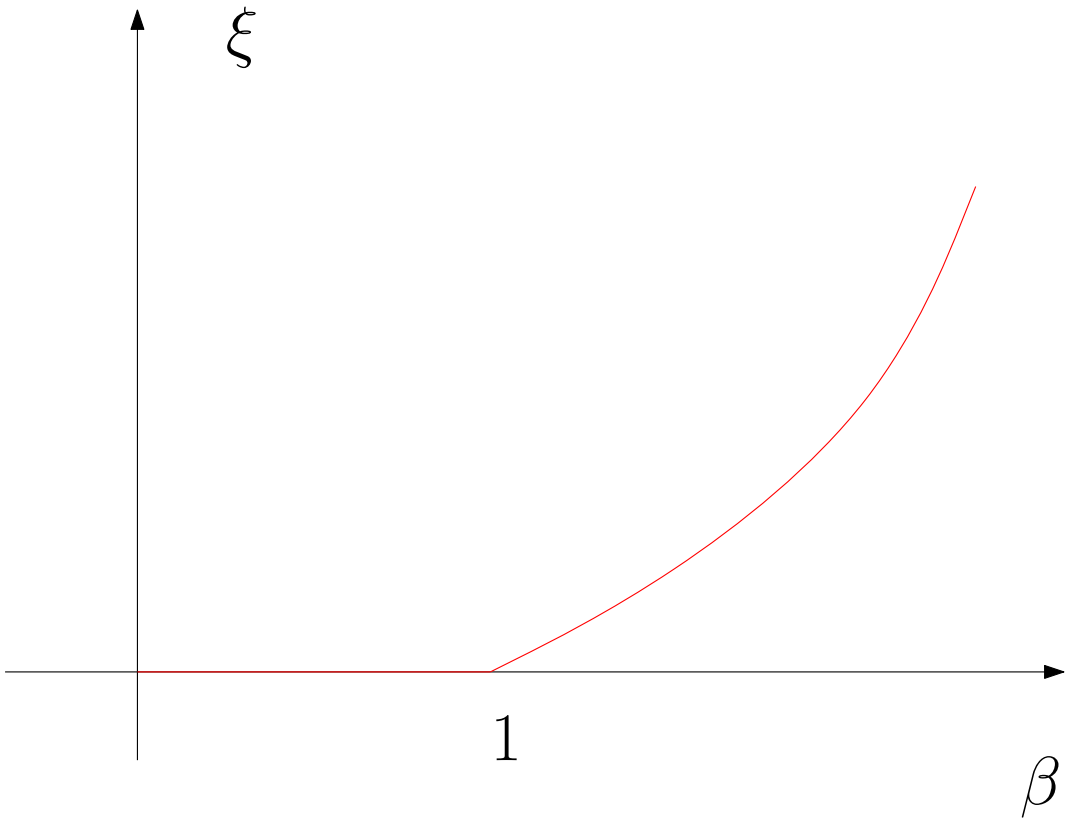

log(cosh(0.5*x))-0.25*x^2
log(cosh(3*x))-3*x^2/2
5
0
-5
-10
-15
-20
-25
-4
-2
0
2
4
x


ξ
1
β

To prove $\mu_{n,\beta} \to_{n\to+\infty} \widetilde{\mu}$ it is sufficient to prove $\lim_{n\to+\infty} \mu_{n,\beta}([\omega]) = \widetilde{\mu}([\omega])$ for any cylinder ω.

We will prove :

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A more general result : Curie Weiss Potts model

Holds for $\Lambda = \{\theta^1, \ldots, \theta^q\}$ with $q > 2$.

$$H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} \mathbb{1}_{\omega_j = \omega_i}. \tag{7}$$

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Theorem (Weak convergence for the CWP model)

For $1 \leq k \leq q$, $b \in \mathbb{R}$, let $\widetilde{\mu}_b^k$ be the dynamical Gibbs measure for $b.\mathbb{1}_{[\theta^k]}$. Let $\beta_c = \frac{2(q-1)\log(q-1)}{q-2}$. For $0 < \beta < \beta_c$ set $s_\beta = 0$ and for $\beta \geq \beta_c$ let s_β be the largest solution of the equation

$$s = \frac{e^{\beta s} - 1}{e^{\beta s} + q - 1}. \tag{9}$$

Then,

$$\mu_{n,\beta} \longrightarrow_{n \to +\infty} \begin{cases} \rho^{\otimes \mathbb{N}} = \widetilde{\mu}_0 & \text{if } 0 < \beta < \beta_c, \\ \dfrac{1}{q} \displaystyle\sum_{k=1}^{q} \widetilde{\mu}^k_{\beta.s_\beta} & \text{if } \beta > \beta_c, \\ \dfrac{A\,\widetilde{\mu}_0 + B \sum_{k=1}^{q} \widetilde{\mu}^k_{\beta_c.s_{\beta_c}}}{A + qB} & \text{if } \beta = \beta_c, \end{cases} \tag{10}$$

Construction of dynamical (local) equilibrium states

Recall : Gibbs measure $\forall x \in \Sigma$, $\forall n$,

$$e^{-C} \leq \frac{\widetilde{\mu}([x_0 \dots x_{n-1}])}{e^{S_n(\phi)(x) - n\mathcal{P}}} \leq e^{C}.$$

Pressure + equilibrium state :

$$\mathcal{P}(\beta.\phi) := \sup_{\mu} \left\{ h_\mu + \beta \int \phi \, d\mu \right\}.$$

Ruelle-Griffiths theorem : If ϕ is Lipschitz continuous unique equilibrium state= Gibbs state + $P(\beta)$ analytic.

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Definition

Transfer operator for potential A,

$$\begin{aligned}\mathcal{L}_A(\varphi)(x_0x_1\cdots) &= \sum_{y,\sigma(y)=x} e^{A(y)}\,\varphi(y)\\ &= \sum_{a,ax_0\in\mathcal{A}} e^{A(ax_0x_1x_2,\ldots)}\,\varphi(ax_0x_1x_2\ldots).\end{aligned}$$

Also called Ruelle-Perron-Frobenius operator.

1. $\mathcal{L}_A$ acts on continuous and Lipschitz functions.
2. Spectral radius λ_A is an eigenvalue for $\mathcal{L}_A$ and $\mathcal{L}_A^*$.
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Spectral Gap $\Longrightarrow$ Analyticity.

If A only depends on 2 coordinates $A(x_0 x_1 x_2 \ldots) = A(x_0, x_1)$, then $\mathcal{L}_A$ is a 2×2 matrix.

Theorem

(Perron-Frobenius) *Let $B = (b_{ij})$ be a $d \times d$ matrix with positive entries. Then, the spectral radius of B, say λ, is a simple dominated eigenvalue. The associated eigenspace is generated by some "positive" vector $u = (u_1, \cdots, u_d)$ with $u_i > 0$.*

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Let $M = (M_{ij})$ be the matrix with entries $e^{A(i,j)}$. Let $\mathbf{r} = (r_1, \dots r_d)$ be the right-eigenvector associated to λ with normalization $\sum r_i = 1$. Let $\mathbf{l} = (l_1, \dots, l_d)$ be the left-eigenvector for λ with renormalization $\sum l_i r_i = 1$. Then, $\mathbf{r}$ is the eigenmeasure ν_A and $\mathbf{l}$ is the density H_A.
The Gibbs measure of the cylinder $[i_0 \dots i_{n-1}]$ is $\mu_A = ([i_0 \dots i_{n-1}]) = l_{i_0} e^{S_n(A)(x) - n \log \lambda_A} r_{i_{n-1}}$

(freezing) phase transition ?

Definition

We say that there is a transition phase at β_0 if the pressure function $\mathcal{P}(\beta)$ is not analytic at β_0. We say that it is a *freezing phase transition* if $\mathcal{P}(\beta)$ is affine for every $\beta > \beta_c$.

In case of a freezing phase transition, after transition

$$\mathcal{P}(\beta) = h + \beta.a,$$

with $a = \max \int A \, d\mu$.

In ergodic theory, a measure satisfying this last property is said to be *A-maximizing*. In Statistical Mechanics it is a *ground state*.
$h =$ *residual entropy*=max of h_μ, μ is A-maximizing.

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For Curie Weiss model phase transition means more than one "ergodic" components for $\lim_{n\to+\infty} \mu_{n,\beta}$. Here, phase transition $\perp$ # of equilibrium states.

Examples with $\mathcal{P}(\beta)$ analytic but several equilibirium and examples with $\mathcal{P}(\beta)$ non-analytici but a unique equilbirium.

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How to detect/get freezing phase transition ?

Via an inducing scheme. We consider a cylinder $J = [w_J]$ in Σ. For x in J,

- the first return time is $\tau(x) = \min\{n \geq 1, \sigma^n(x) \in J\} \leq +\infty$.
- First return map $F(x) := \sigma^{\tau(x)}(x)$. This map is well defined if $\tau(x) < +\infty$.
- The main important point is that the inverse branches are well defined everywhere on J

Theorem (Dowker)

If $\widehat{\mu}$ is a σ-invariant probability measure with $\widehat{\mu}(J) > 0$, then, the conditional measure $\mu := \dfrac{\widehat{\mu}(. \cap J)}{\widehat{\mu}(J)}$ is F-invariant.

Conversely, if μ is a F-invariant probability measure and $\displaystyle\int \tau \, d\mu < +\infty$, then there exists a unique σ-invariant probability $\widehat{\mu}$ such that $\mu := \dfrac{\widehat{\mu}(. \cap J)}{\widehat{\mu}(J)}$ holds

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At that stage we have two different dynamical systems : (Σ, σ) and (J, F).

By the Abramov formula

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$$\mathcal{L}_{Z,\beta}(g)(x) = \sum_{n \in \mathbb{N}} \sum_{\substack{\tau(y)=n \\ \sigma^n(y)=x}} e^{\beta . S_n(A)(y) - nZ} g(y)$$

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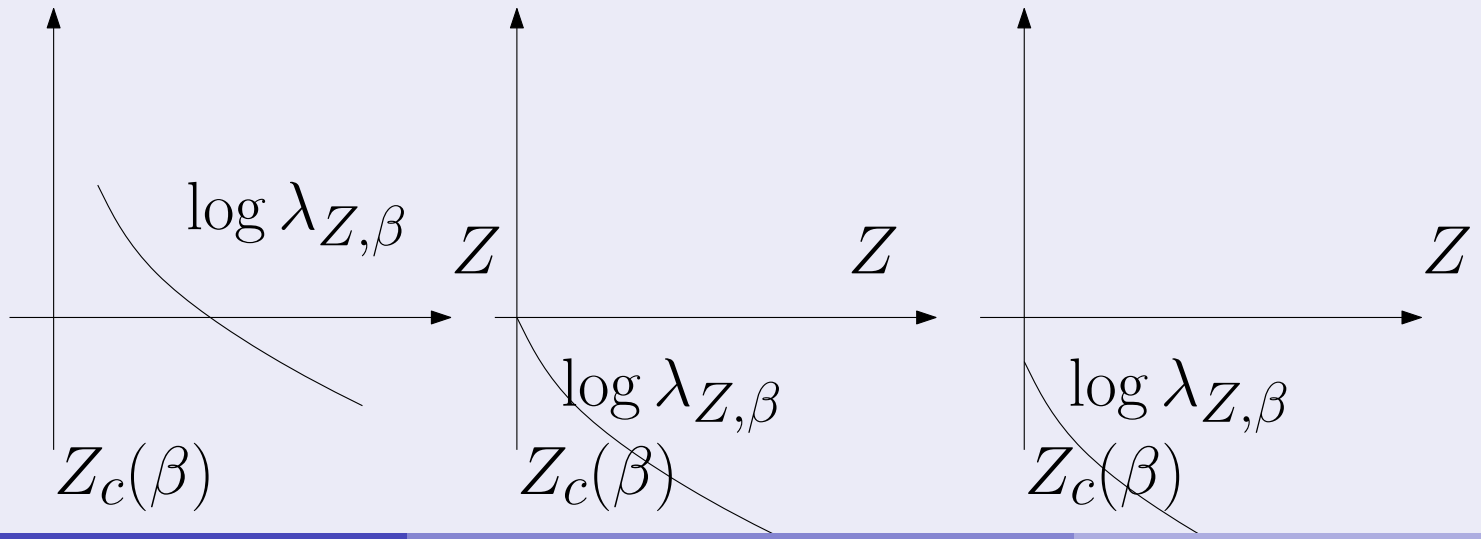

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My lectures in cartoons

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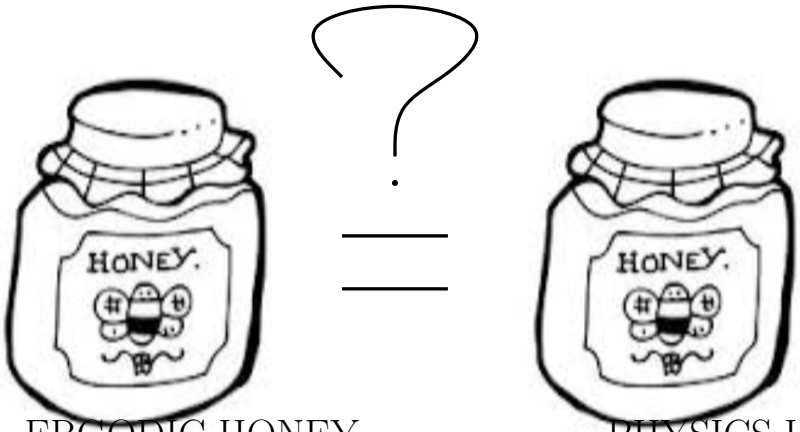
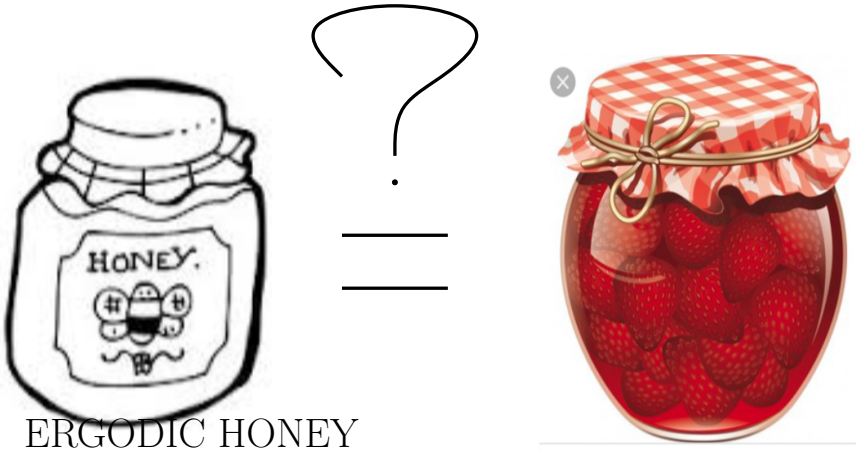


ERGODIC HONEY PHYSICS HONEY

Ergodic Gibbs Measure $\approx$ Limit of Gibbs measures for Curie-Weiss Model



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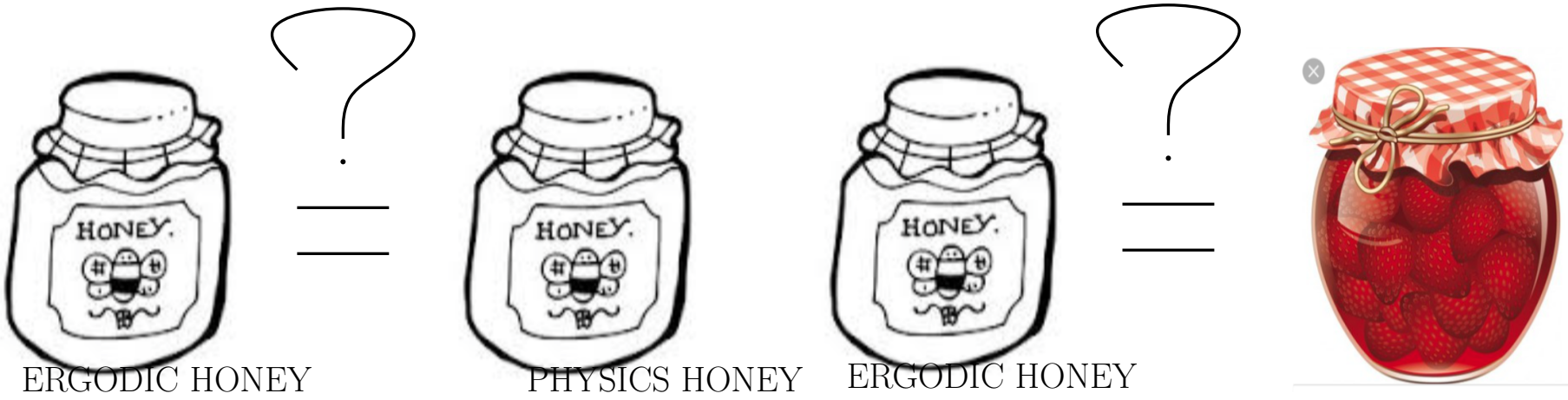

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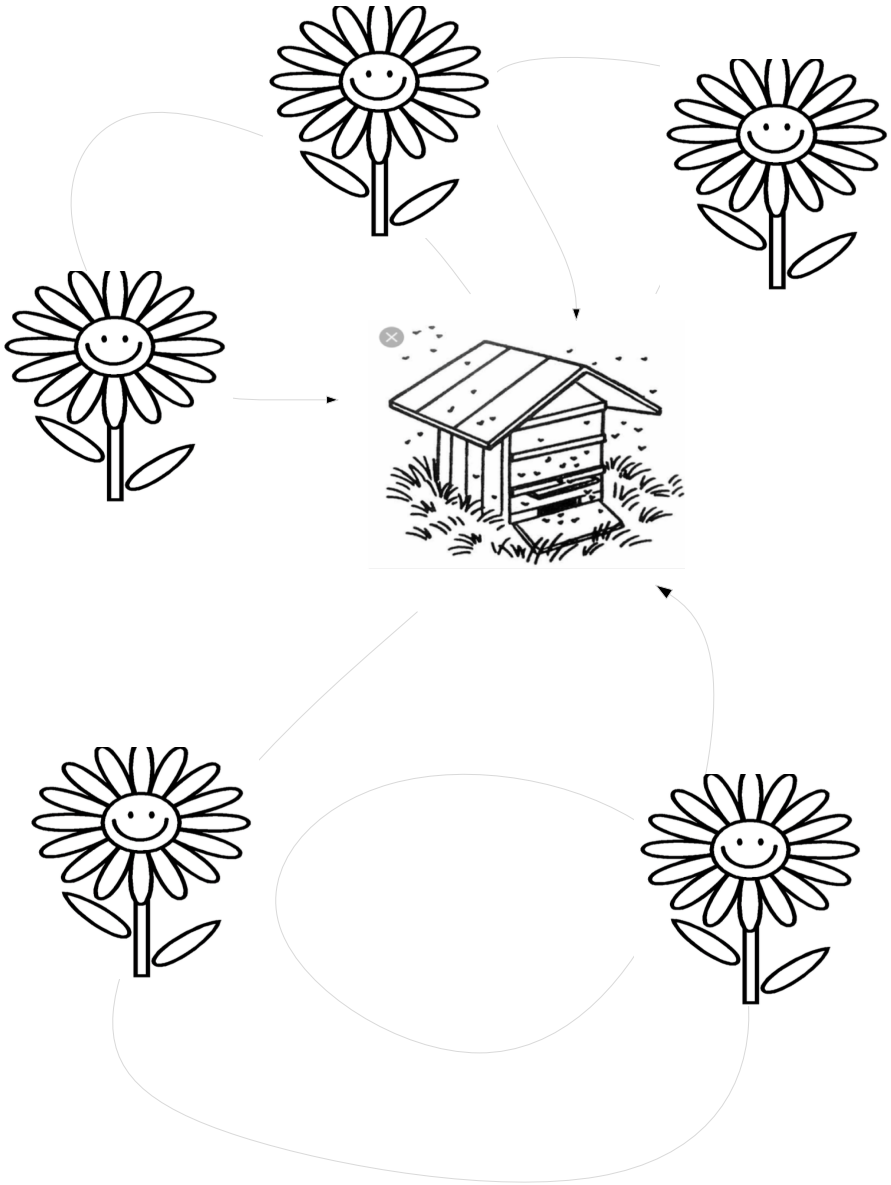

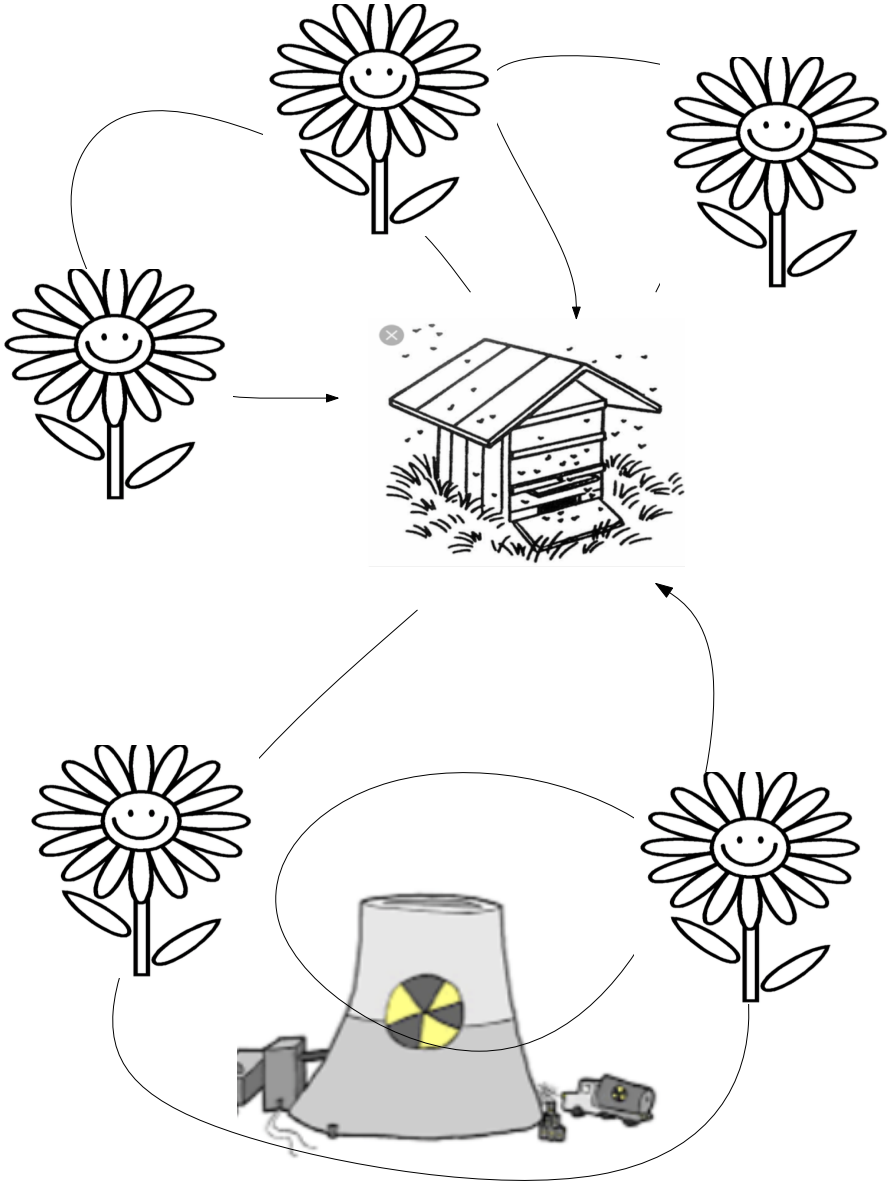

Critical level

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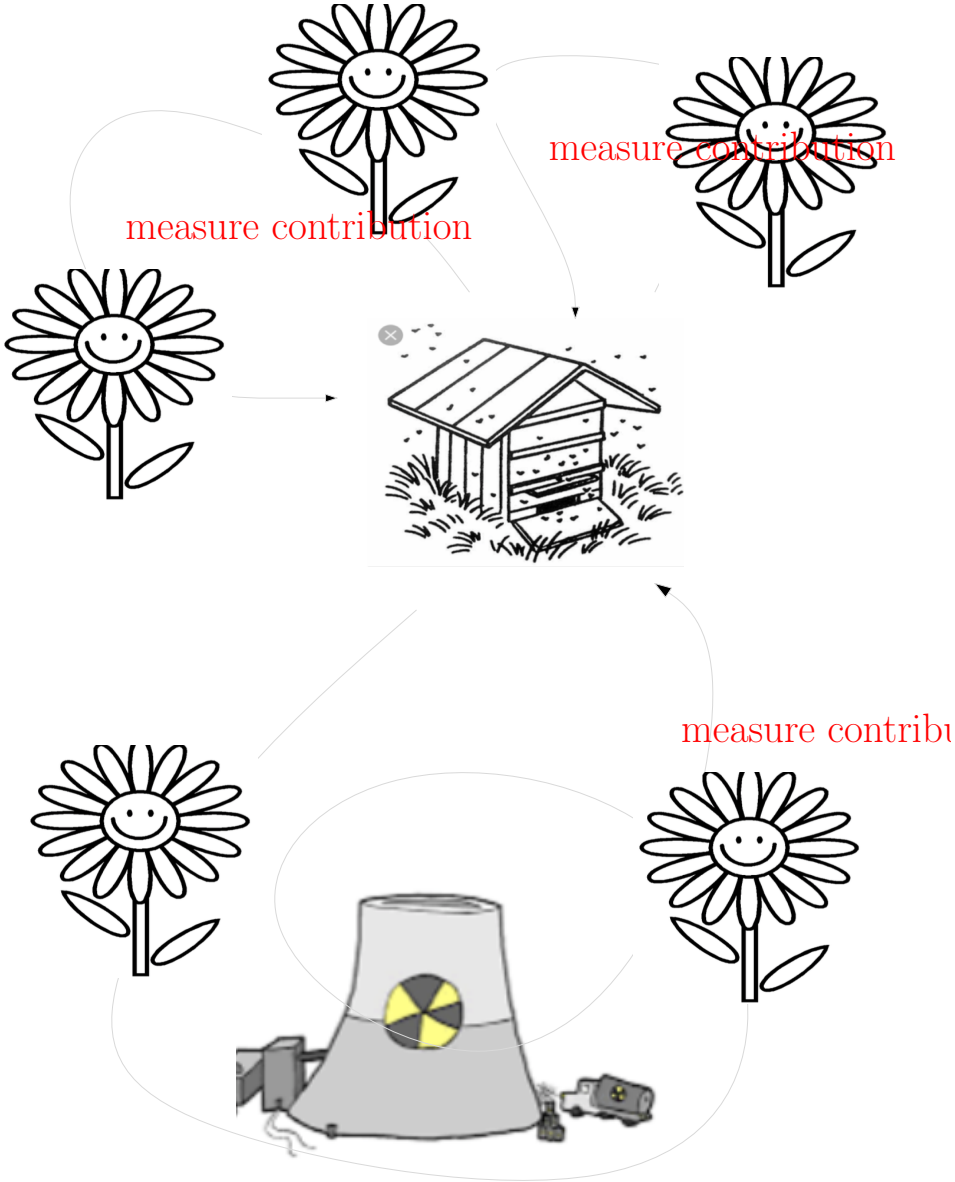
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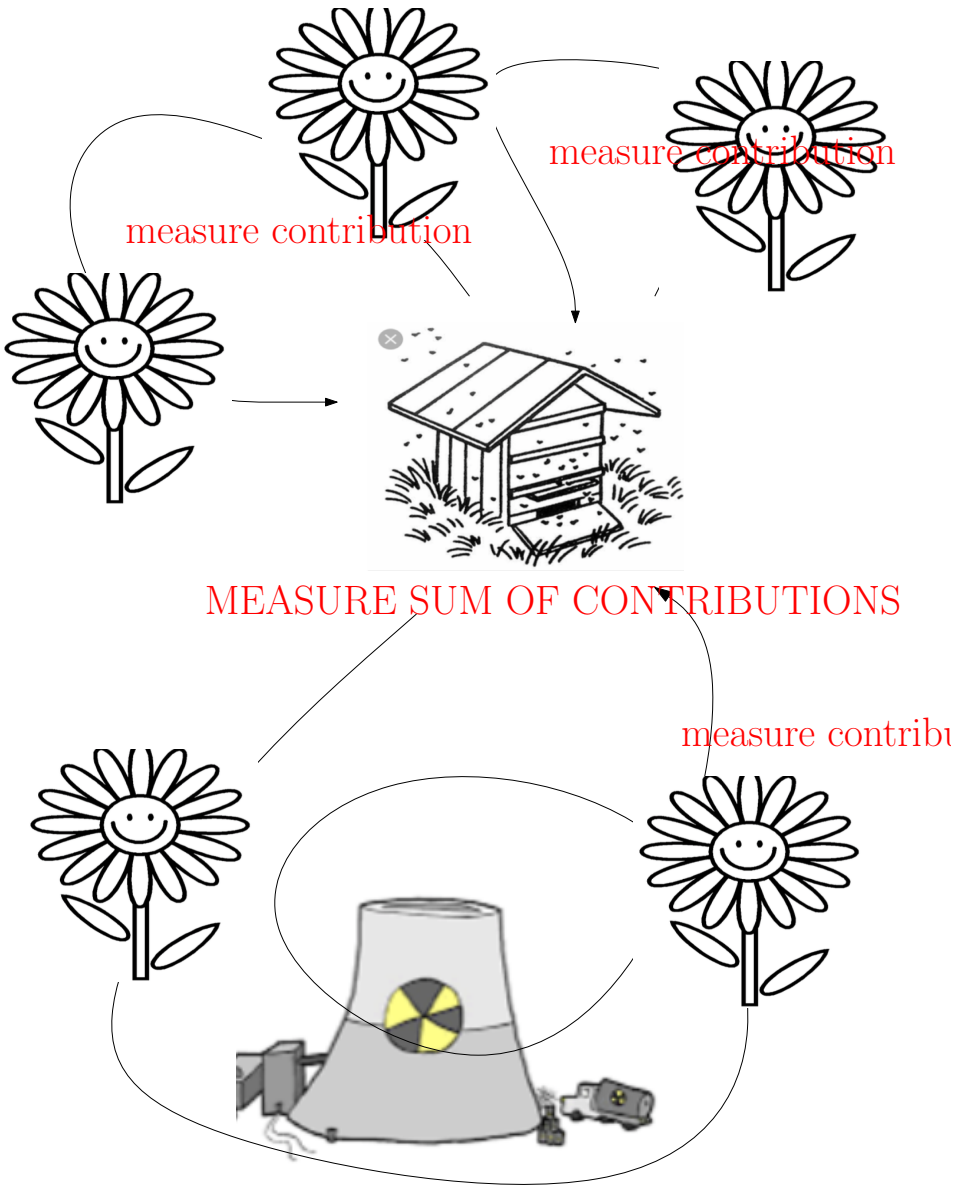
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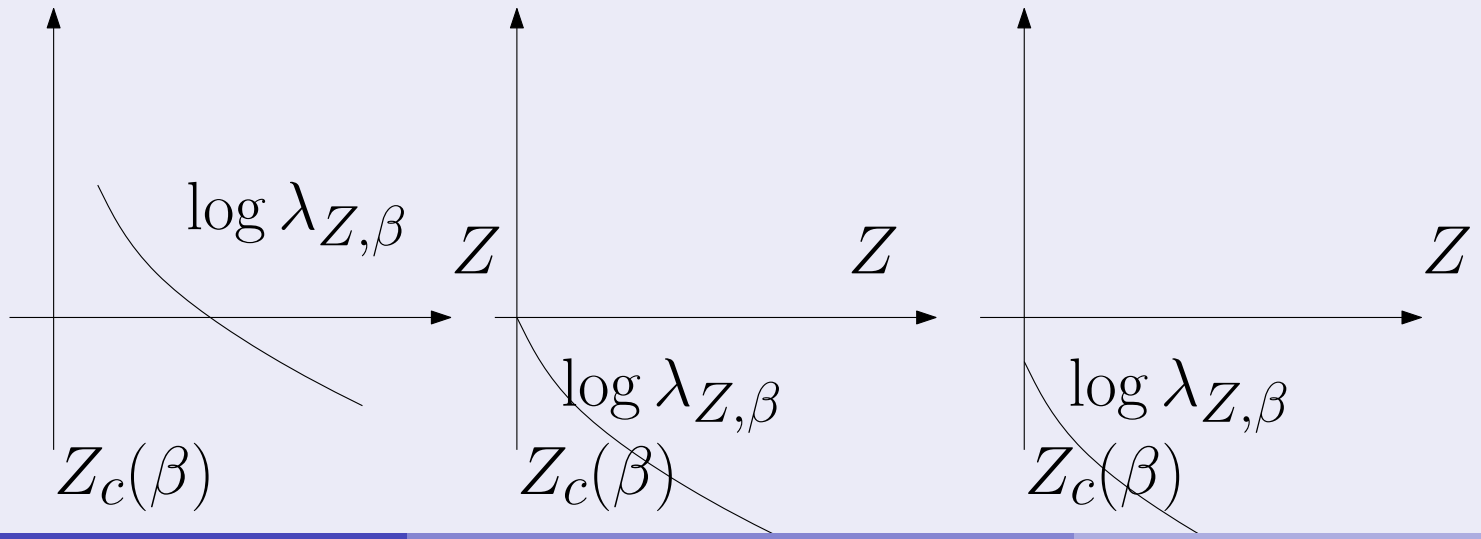

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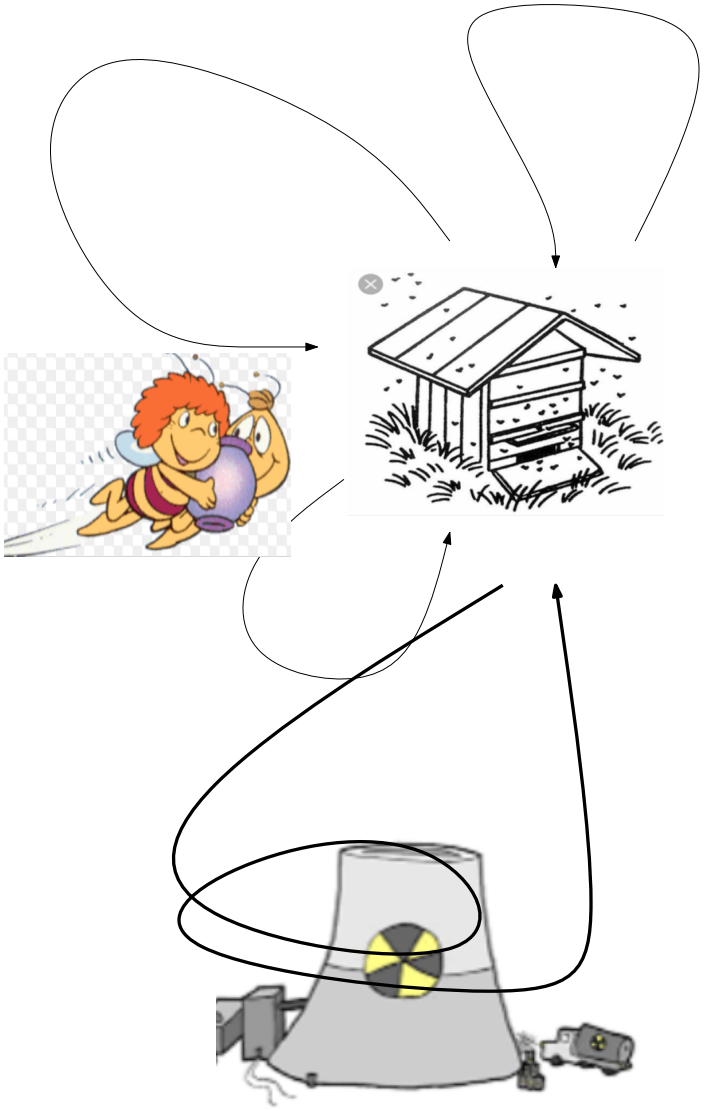

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$\approx \Phi$ (potential)

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There exist sophisticated baskets such that if β is sufficiently big (temperature sufficiently small) the bees do no collect to much contaminated pollen.

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Consider a primitive, aperiodic and marked substitution, $\mathbb{K}$ the attractor associated to H and consider the potential defined by $V(x) := -\log\left(\frac{n+1}{n}\right)$ if $d(x,\mathbb{K}) = 2^{-n}$. Then there exists a positive number β_c such that the pressure function has a freezing phase transition at β_c. More precisely :

- *For $\beta < \beta_c$ the pressure function is analytic, there is a unique equilibrium state for $\beta.V$ and it has full support.*
- *For $\beta > \beta_c$ the pressure is equal to zero and $\mu_{\mathbb{K}}$ is the unique equilibrium state for $\beta.V$.*