Symmetries of monocoronal tilings

based on a joint work with Dirk Frettlöh

Alexey Garber

The University of Texas Rio Grande Valley

June 7, 2016
Oléron

Tilings

Definition

A collection $\mathcal{T}$ of (convex) polytopes in $\mathbb{R}^d$ is called *(locally finite) tiling* if

- union of all polytopes from $\mathcal{T}$ is $\mathbb{R}^d$;
- they do not intersect in internal points;
- every ball intersects only finite number of polytopes from $\mathcal{T}$.

TILINGS

Definition

A collection $\mathcal{T}$ of (convex) polytopes in $\mathbb{R}^d$ is called *(locally finite) tiling* if

- union of all polytopes from $\mathcal{T}$ is $\mathbb{R}^d$;
- they do not intersect in internal points;
- every ball intersects only finite number of polytopes from $\mathcal{T}$.

Definition

A tiling is called *face-to-face* or *normal* if intersection of any two tiles is a face of both.

The question

Is it true that unique local structure of a tiling $\mathcal{T}$ implies that $\mathcal{T}$ posesses a "rich" symmetry group?

Periodic, k-periodic, and non-periodic tilings

Definition

A tiling of $\mathbb{R}^d$ is called *periodic* if it has d-dimensional translation group.

Definition

A tiling of $\mathbb{R}^d$ is called *k-periodic* (with $k < d$) if it has k-dimensional translation group. If a tiling is 0-periodic then it is also called a *non-periodic* tiling.

CRYSTALLOGRAPHIC TILINGS

Definition

A tiling $\mathcal{T}$ is called *crystallographic* if its symmetry group has a compact fundamental domain.

In Euclidean space crystallographic tiling is the same as periodic tiling, but this definition can be used in the hyperbolic space $\mathbb{H}^d$ too.

Coronae of a tile

Definition

The *k-th facet corona* of an arbitrary tile P is the collection of all tiles of $\mathcal{T}$ that can be reached from P by at most k steps across facets of $\mathcal{T}$.

Theorem (Generalized Local Theorem by N. Dolbilin and M. Shtogrin)

A tiling of $\mathbb{R}^d$ (or $\mathbb{H}^d$) is crystallographic iff for some k the following conditions hold.

- *For the number $N(k)$ of k-th facet coronae we have: $N(k) = N(k+1)$ and this number is finite.*
- *For every i the symmetry groups $S_i(k)$ and $S_i(k+1)$ of k-corona and $(k+1)$-corona of the i-th type coincide.*

Vertex corona

Consider an arbitrary vertex A of the tiling $\mathcal{T}$.

Definition

The set of all polytopes contains A is called *the vertex corona* of A

Definition

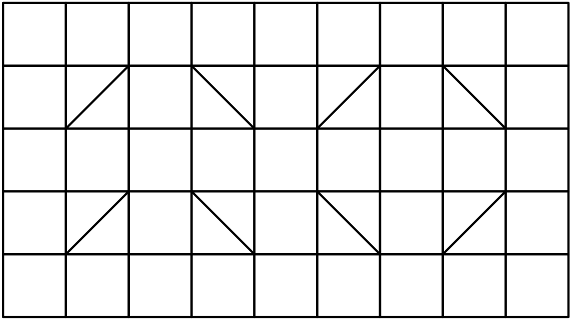
A tiling $\mathcal{T}$ is said to be a *monocoronal* if all its vertex coronae are congruent. This means not only collections of polytopes are the same but also that they arranged at corresponding vertices in the same way.

Unique vertex corona and periodicity

Question

Is it true that every monocoronal tiling $\mathcal{T}$ is periodic (or crystallographic)?

Why do we need to use only one corona?



Idea of face-to-face classification: combinatorics

Lemma

If every vertex corona contains n polygons with sides $a_1, a_2, \ldots, a_n$ then

$$\sum \frac{1}{a_i} = \frac{n-2}{2}.$$

Idea: count the average angle that a polygon brings to a neighborhood of a vertex.

There are finitely many integer solutions of this equation (however not every solution can be realized as a monocoronal tiling).

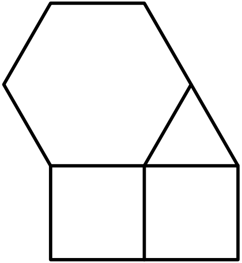
Idea of face-to-face classification: combinatorics

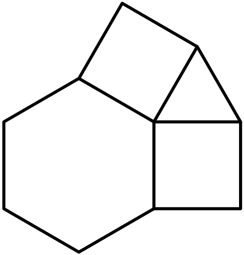
For example, there is a solution $a_1 = 1, a_2 = a_3 = 4, a_4 = 6.$

Idea of face-to-face classification: combinatorics

For example, there is a solution $a_1 = 1, a_2 = a_3 = 4, a_4 = 6$.

Two theoretical local structures are

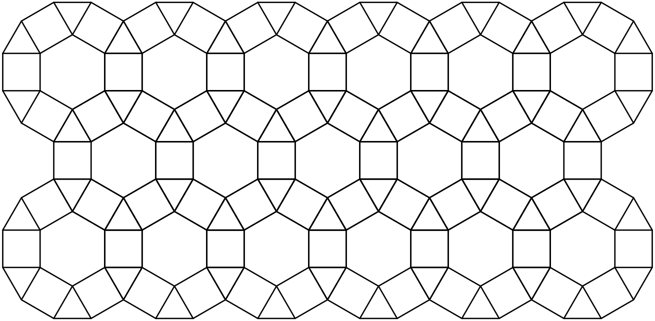




Idea of face-to-face classification: combinatorics

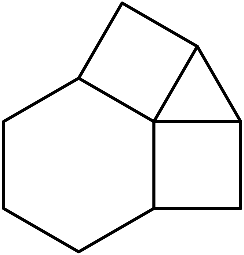
For example, there is a solution $a_1 = 1, a_2 = a_3 = 4, a_4 = 6$.

The only possible combinatorial structure is the following:



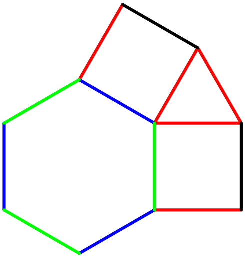
Idea of face-to-face classification: metric properties

- We mark segments that are equal with the same color.



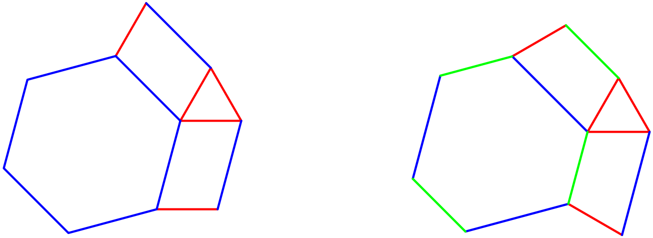
Idea of face-to-face classification: metric properties

- We mark segments that are equal with the same color.



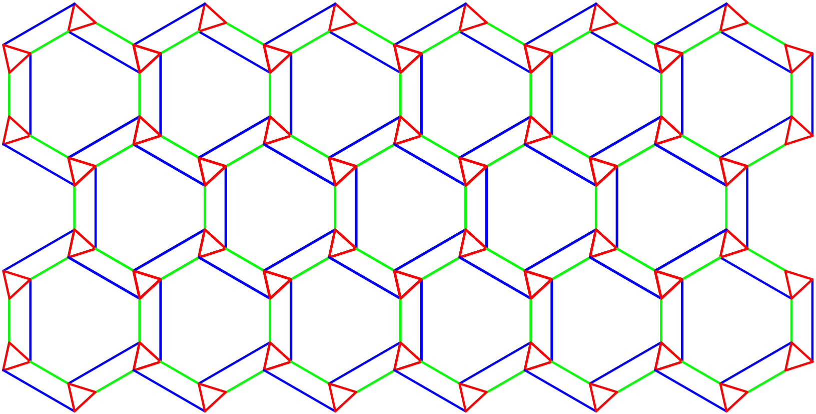
Idea of face-to-face classification: metric properties

- We mark segments that are equal with the same color.

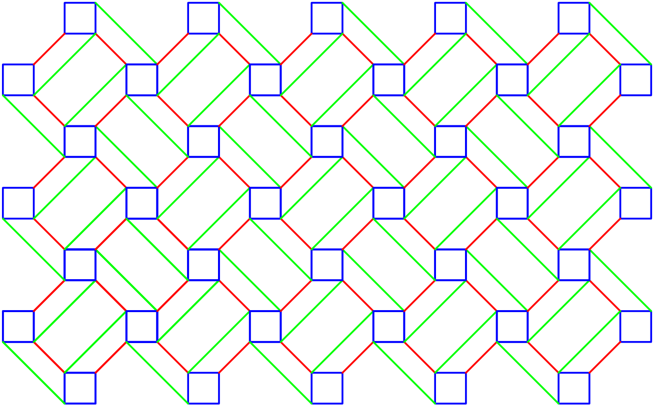


In this particular case there are two possibilities.

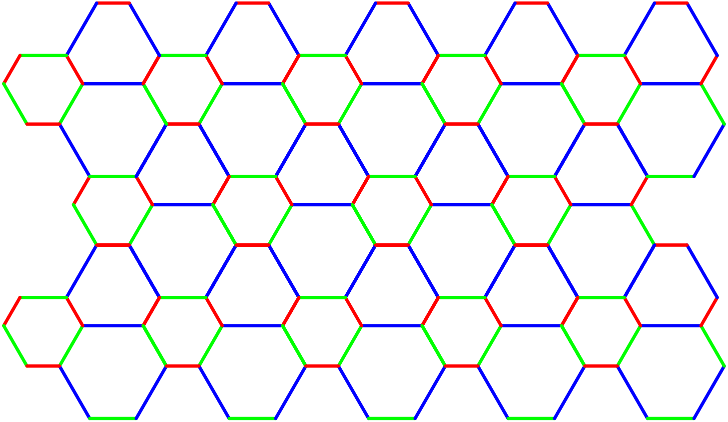
Some examples of monocoronal tilings



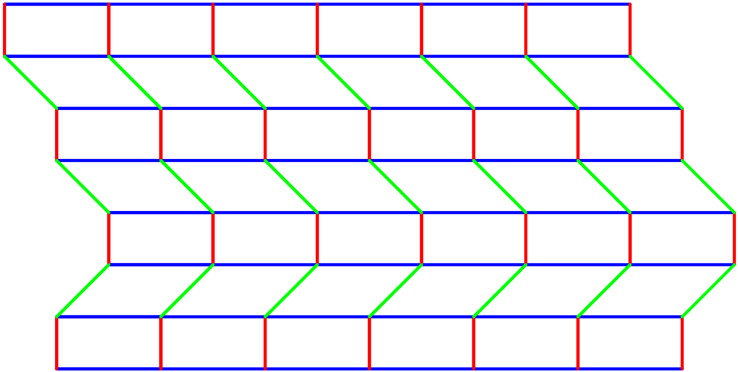
Some examples of monocoronal tilings



Some examples of monocoronal tilings



Example of a 1-periodic tiling

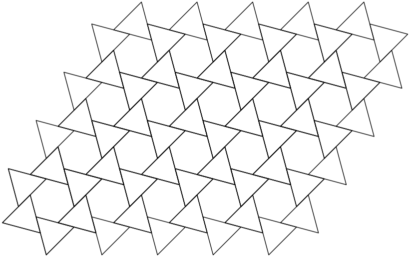


Idea of non face-to-face classification

Lemma

Assume every vertex corona of A contains $n+1$ polygons one of which contains vertex on its side. Let other polygons have $a_1, a_2, \ldots, a_n$ sides. Then

$$\sum \frac{1}{a_i} = \frac{n-1}{2}.$$



Properties of two-dimensional monocoronal tilings

Claim

A two-dimensional monocoronal tiling $\mathcal{T}$ is not necessary periodic.

Claim

$\mathcal{T}$ has at least one-dimensional translation group $G_{\mathcal{T}}$.

Claim

If $G_{\mathcal{T}}$ is one-dimensional then we must use corona $C_{\mathcal{T}}$ and its reflected image (rotations are not enough).

Theorem (D. Frettlöh, A.G.)

16 of 17 crystallographic groups and 4 of 7 frieze groups can appear as a symmetry group of a monocoronal tiling.

Further questions about monocoronal tilings in arbitrary dimensions

Question

What is the minimal dimension of the translation group $G_{\mathcal{T}}$ a monocoronal can have?

Question

Can a monocoronal tiling with unique "non-reflected" vertex corona be non-periodic?

Face-to-face tilings

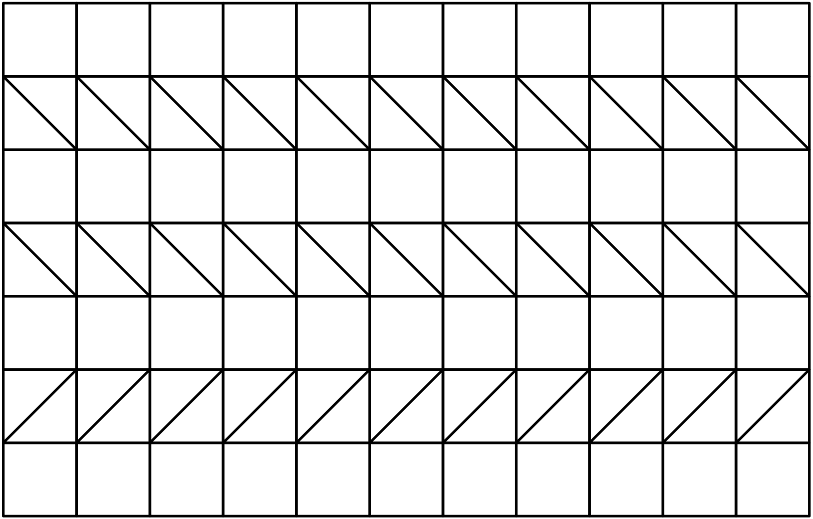
Theorem (D. Frettlöh, A.G.)

There are d-dimensional face-to-face monocoronal tilings with translation group of dimension $\lceil \frac{d}{2} \rceil$.

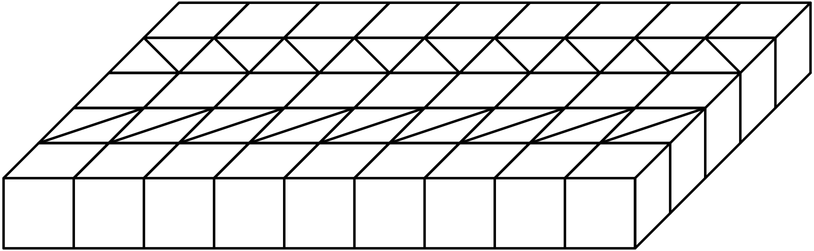
Theorem (D. Frettlöh, A.G.)

There are d-dimensional face-to-face monocoronal tilings with directly congruent coronae (rigid motions only!) with translation group of dimension $\lceil \frac{d+1}{2} \rceil$.

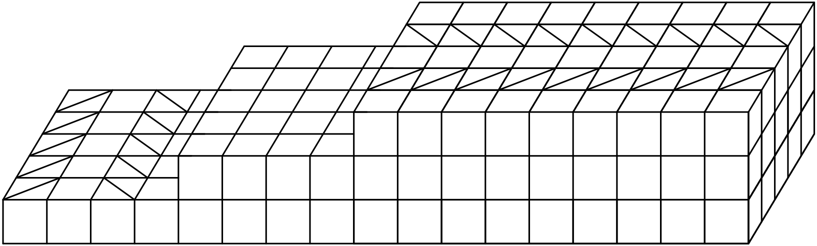
Non-face-to-face tilings



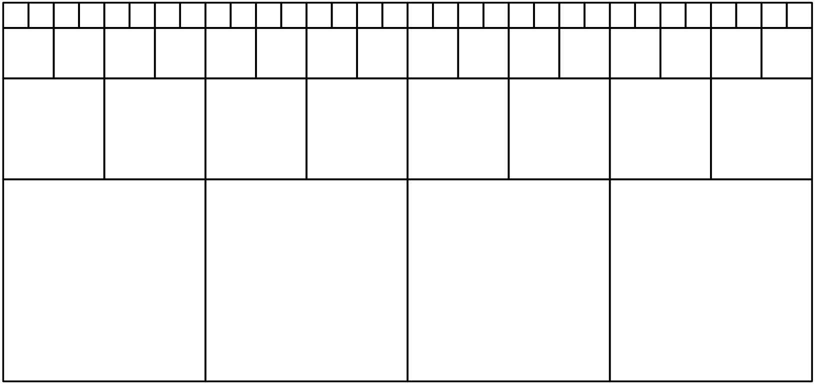
Non-face-to-face tilings



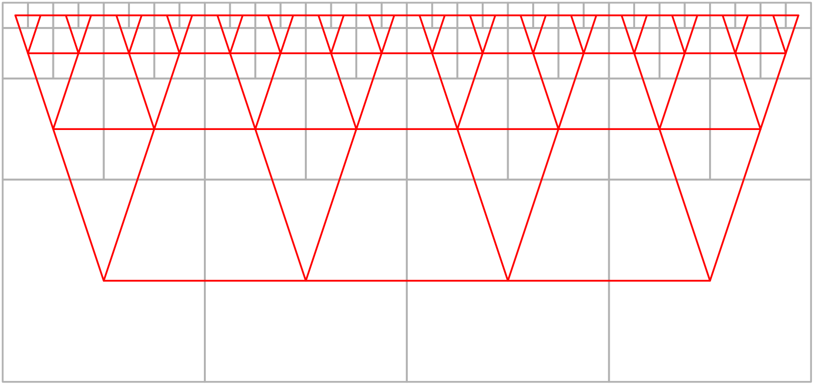
Non-face-to-face tilings



Böröczky tiling



Dual tiling



Open question

Question

What is the minimal dimension the translation group of a d-dimensional monocoronal face-to-face tiling?

In particular, can a three-dimensional monocoronal face-to-face tiling be non-periodic?

THANK YOU!