Strongly Aperiodic Subshifts of Finite Type on Surface Groups

Chaim Goodman-Strauss, U. Arkansas
joint with David Cohen, U. Chicago

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We give strongly aperiodic subshifts of finite type on every hyperbolic surface group; more generally, for each pair of expansive primitive symbolic substitution systems with incommensurate growth rates, we construct strongly aperiodic subshifts of finite type on their orbit graphs.

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- $\mathbb{Z}^2 \rtimes G$ for any group with decidable word problem (Barbieri and Sablik, 2016)

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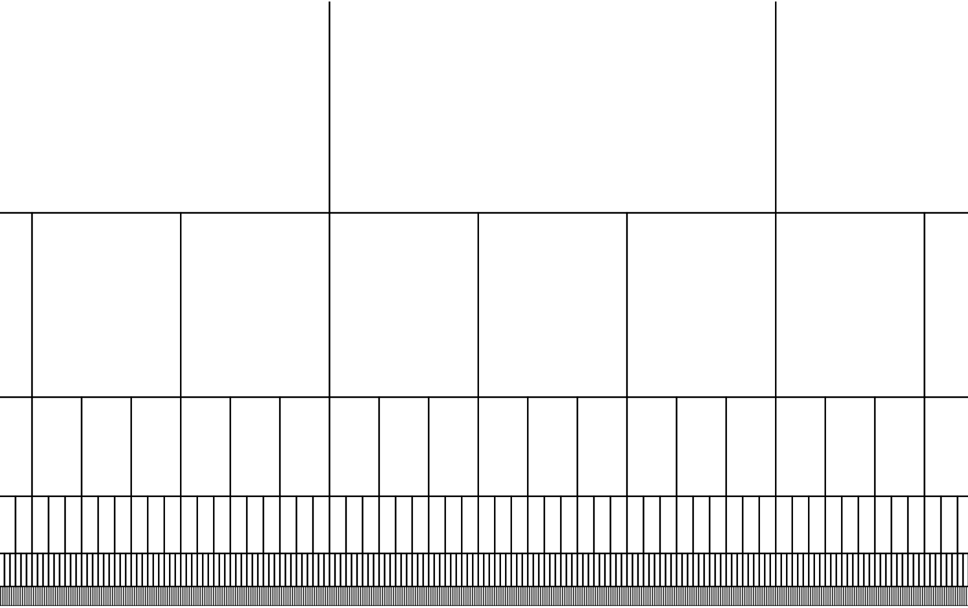
nor, remarkably, does any group with undecidable word problem (Jeandel, 2015).

(It is not too hard, typically, to construct “weakly aperiodic" subshifts of finite type, in which there exist elements stabilized by $\mathbb{Z}$, but none with co-finite stabilizer.)



Though *strongly* aperiodic tilings have been constructed in $\mathbb{H}^2$, the underlying tiles do not admit a tiling of any compact fundamental domain, and do not directly shed light on SASFT's for hyperbolic surface groups.

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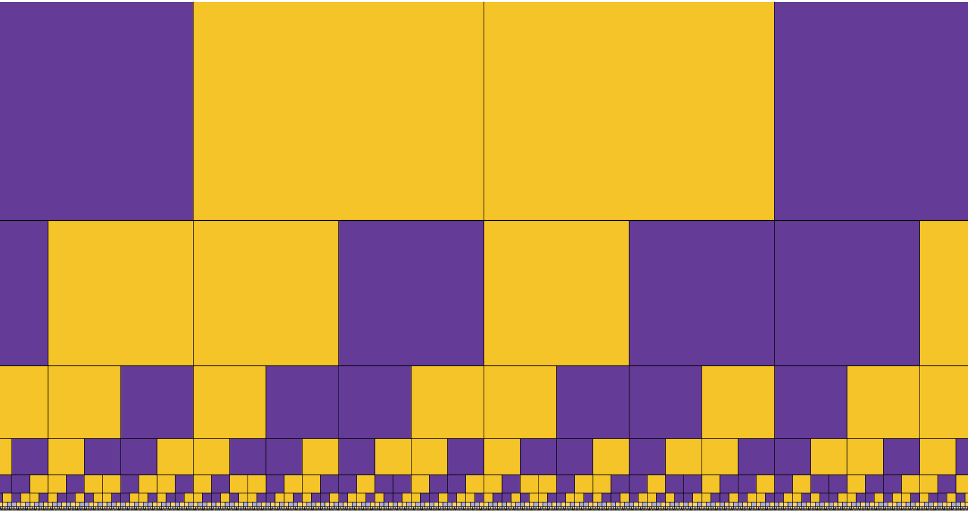
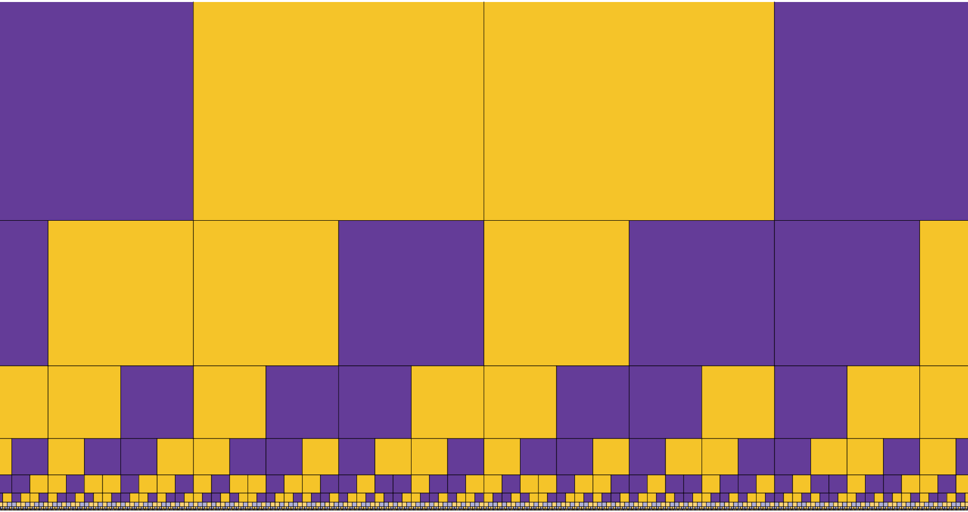
- Surface groups can be realized in primitive symbolic substitution systems.





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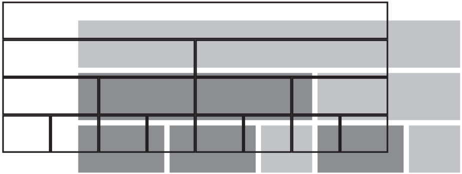

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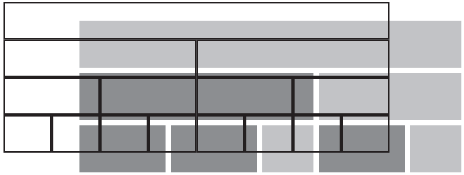

Here we show $\mathtt{0} \mapsto \mathtt{0}^2$ versus $\mathtt{a} \mapsto \mathtt{ab}, \mathtt{b} \mapsto \mathtt{aab}$ The critical observation is that these relative growth rates are highly constrained. In effect, we create a set of tiles encoding the local combinatorics of how these tilings may meet; any tiling by these tiles must enforce the growth rates of the underlying substitution systems. In turn, as these growth rates are incommensurate, there can be no vertical period.

(This essentially a generalization of the construction in (GS 2005), in turn derived from (Kari 1996)— in our terms here, the underlying substitution systems there are simply $\mathtt{0} \mapsto \mathtt{0}^2$ and $\mathtt{0} \mapsto \mathtt{0}^3$.)

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- Cayley graphs of surface groups can be embedded in orbit graphs of primitive symbolic substitution systems.
- Finally, we pull back a SASFT on such an orbit graph to the surface group.