Groups of intermediate growth
and
aperiodic order

Rostislav Grigorchuk
Texas A&M University

Oleron, France

1 Two examples

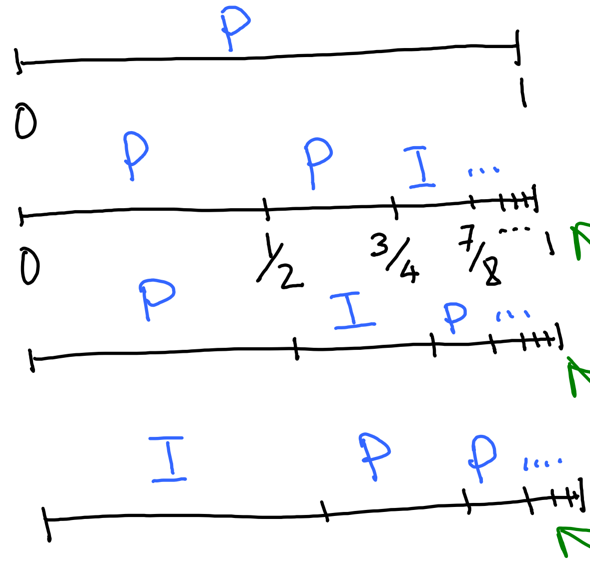
Gri. 1980

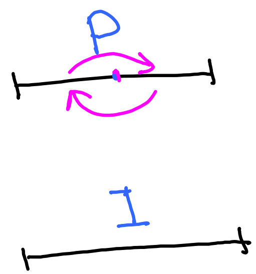
a

b

c

d





permutation

identity

$(PPI)^\infty$

$(PIP)^\infty$

$(IPP)^\infty$

periodic sequences

$a^2 = b^2 = c^2 = d^2 = id$

four involutions

action by Lebesgue measure preserv. tr.

$$\mathcal{G} = \langle a, b, c, d \rangle \curvearrowright [0,1] \setminus \left\{ \frac{m}{2^n}, m, n \in \mathbb{Z} \right\}$$

group generated by a, b, c, d

$\langle b, c, d \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ – Klein 4-group

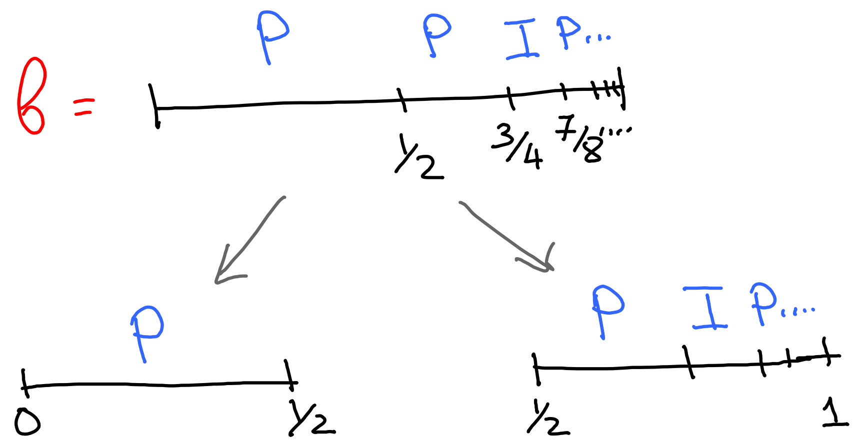
$\quad\; \shortparallel$

$\{1, b, c, d\}$

Property 1. $\mathcal{G}$ is self-similar.

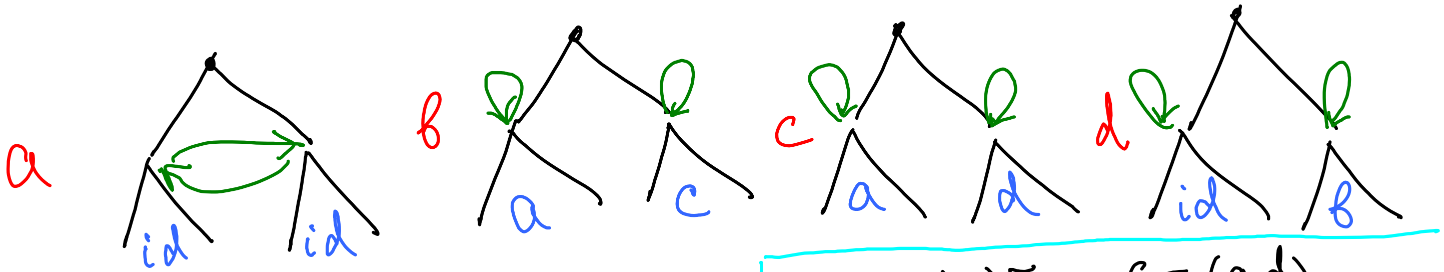
Example $\quad b =$

$a =$ $\qquad\qquad\qquad\qquad = c$



restriction of elements of G on invariant subinterval is again an element of G after identification of subinterval with the whole interval.

Alternative model by action on rooted binary tree T



$id = e = 1$ - identity

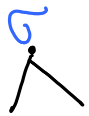
$a = (1,1)\sigma \qquad c = (a,d)$

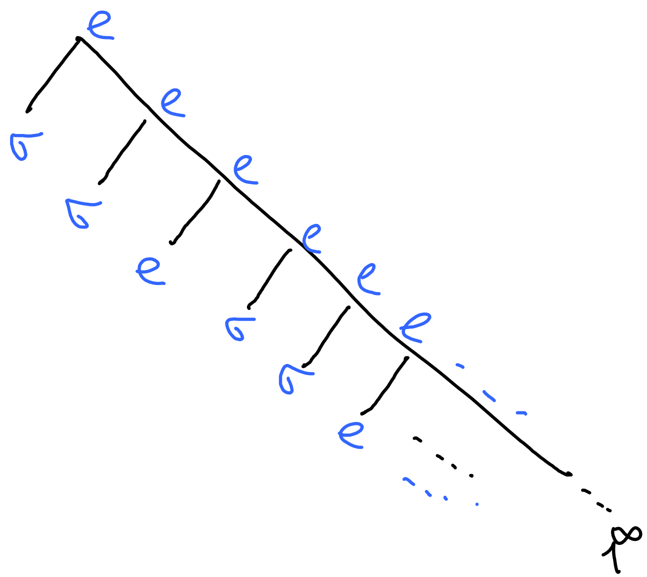
$b = (a,c) \qquad d = (1,b)$

recursive relations

Binary rooted tree

Portraits

$\mathcal{P}(a) =$ 

$\mathcal{P}(b) =$ 

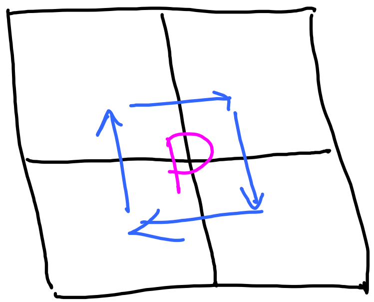
$S_2 = \{e, \sigma\} \curvearrowright \{0, 1\}$

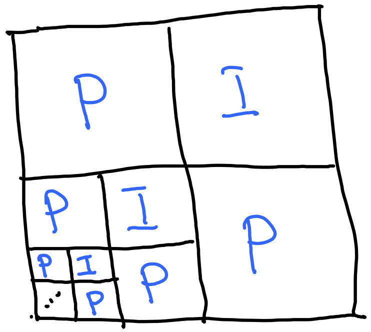
symmetric group

alphabet

$\mathcal{P}(g)$ – labelling of $V(T)$ by elements of symmetric group

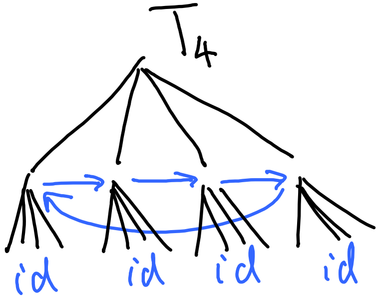
(ii) Second example $\quad \mathcal{G}_1 = \langle a, b \rangle$

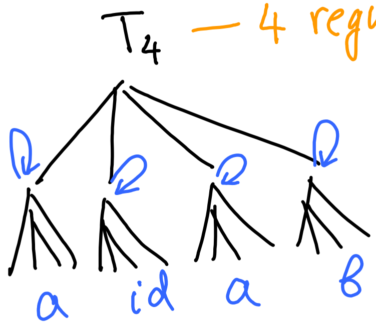
$a =$ 

$b =$ 

P – 4 - cycle

T_4 — 4 regular rooted tree

$a =$ 

$b =$ 

$a = (1,1,1,1)\sigma$, $\sigma \overset{4}{=} 1$ - cycle,

$b = (a, 1, a, b)$ recursive relation

Th. a) 1980 $\mathcal{G}$ and $\mathcal{G}_1$ are finitely generated infinite <u>torsion</u> groups (Burnside groups, 2-groups)

b) 1983 $\mathcal{G}$ is a group of <u>intermediate</u> growth between polynomial and exponential. Moreover

$$e^{n^\alpha} \prec \gamma(n) \prec e^{n^\beta} \qquad (*)$$

where $\alpha = \frac{1}{2}$, $\beta = \log_{32} 31 < 1$.

[Now $(*)$ improved, $\alpha = \frac{1}{2} + 0.04$, $\beta = 0.7574$]

Y. Leonov, L. Bartholdi, P. Muchnik, I. Pak

c) $\mathcal{G}$ and $\mathcal{G}_1$ are amenable but not elementary amenable groups.

For b) this follows from intermediate growth, 1983

For c) this was proved in 2005, Bartholdi, Kaimanovich, Nekrashevych, Bounded automata.

It is still unknown if $\mathcal{G}_1$ has intermediate growth.

Three problems: Burnside, Milnor, Day.

GROWTH

A. Schwarz 1955

J. Milnor 1968

G – finitely generated group, $S = \{s_1, \dots, s_m\}$ – generating set.

$|g| = \min \{ n \mid g = s_{i_1}^{\pm 1} \dots s_{i_n}^{\pm 1} \}$ – length of element g

$\gamma(n) = \gamma_G^S(n) = \#\{ g \in G \mid |g| \leq n \}$ – growth function of G

$\gamma_1(n) \preceq \gamma_2(n) \Longleftrightarrow \exists C$ s.t. $\gamma_1(n) \leq C\gamma_2(n)$

$\gamma_1(n) \sim \gamma_2(n) \Leftrightarrow \gamma_1 \preceq \gamma_2, \ \gamma_2 \preceq \gamma_1.$

equivalence relation

$\boxed{\gamma_G^A(n) \sim \gamma_G^B(n)}$, A, B – finite generating systems

$[\gamma_G(n)]$ – growth degree [rate of growth]

class of equivalence

invariant not only up to isomorphism but even up to quasi-isometry.

$\gamma_G^A(n) = \#(B_e^G(n))$ – ball of radius n around identity, in the

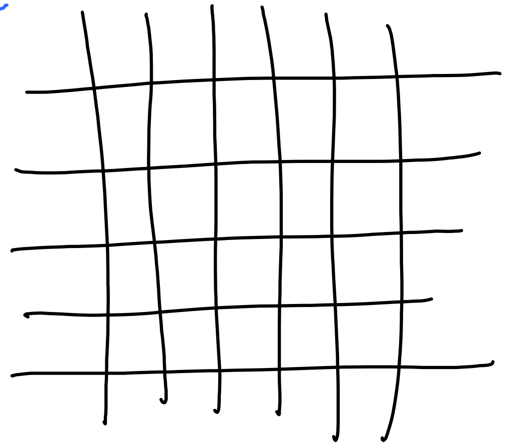
Cayley graph $\Gamma(G, A)$

$V(\Gamma) = G$, $E(\Gamma) = \{(g, ag) : a \in A\}$

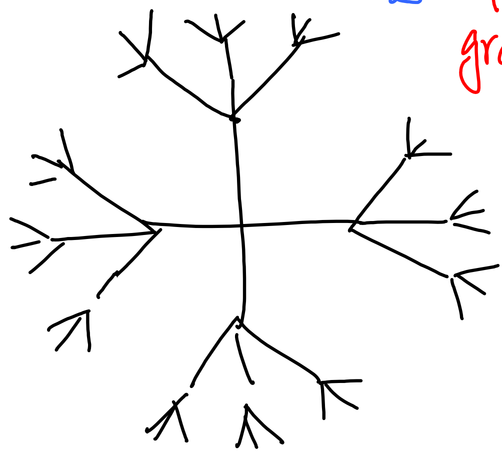
vertices

edges (usually labelled by generators)

$\mathbb{Z}^2$



F_2 - free group of rank 2



Growth can be:

polynomial if $\exists C, d$ s.t. $\forall n \in \mathbb{N}$

$$\gamma_G(n) \le C n^d$$

exponential if $\gamma_G(n) \sim e^n$

Or intermediate $\forall d \quad n^d \underset{\neq}{\prec} \gamma_c(n) \underset{\neq}{\prec} e^n$

Milnor's Conjecture and Question 1968:

(i) Conj. A group has polynomial growth iff it is virtually nilpotent.

Confirmed by M. Gromov 1981

(ii) Question. is it correct that the growth of a group is either polynomial or exponential?

No, $\mathcal{G} = \langle a, b, c, d \rangle$ has intermediate growth

Banach-Tarsky paradox

von Neumann

1929

Amenability ← No Ponzi shemes :)

A discrete group is amenable if it has LIM (left invariant mean)

$\Longleftrightarrow$ many equivalent definitions

Tarsky (paradoxical decompositions), Kesten (spectral radius of random walk), Hulanitsky (weak containment of trivial representation in the regular representation), Gri... (cogrowth criterion), Fölner condition

$$c(\Gamma(G,A)) = 0$$

Cheeger constant

$$c(\Gamma) = \inf_{\substack{F \subset G \\ |F| < \infty}} \frac{|\partial F|}{|F|}$$

Bogolyubov 1939 Topological groups with LIM

Th. G has LIM $\Rightarrow$ for each continuous action by homeomorphisms on a compact space there is a probability G-invariant measure. (generalization of Bogolyubov-Krylov theorem).

$\boxed{\text{Examples}}$

AG - class of (discrete) amenable groups

finite, commutative groups $\in AG$.

AG is closed w.r. 1) taking subgroup,

2) quotient, 3) extentions, 4) inductive limits (unions)

EG - elementary amenable groups (the smallest class of groups containing finite and commutative groups and closed w.r. operations 1) — 4).

nilpotent, solvable, locally finite $\in EG$.

F_2 - simplest example of non-amenable group

$\Rightarrow$ if G contains F_2 as subgroup then $G \notin AG$.

NF - class of groups with no subgroups $\simeq F_2$.

M. Day 1957 Two questions.

a) $AG \overset{?}{=} NF$ NO $\leftarrow$ Olshanski 1980, used cogrowth criterion and Tarsky monsters

b) $AG \overset{?}{=} EG$ NO, Gri, 1983, used solution of Milnor's question.

$\boxed{\mathcal{G} \in AG \setminus EG}$

Current situation

$$EG \subsetneq SG \subsetneq AG \subsetneq NF$$

class of subexponentially amenable group (the smallest class of groups containing groups of subexponential growth and closed w.r. 1) – 4)).

$B \in AG \setminus SG$

Basilica group $=$ $IMG(z^2-1)$ has exponential growth

iterated monodromy group

$IMG(z^2+i)$ has intermediate growth $\Rightarrow \in AG \setminus EG$

Bux, Perez

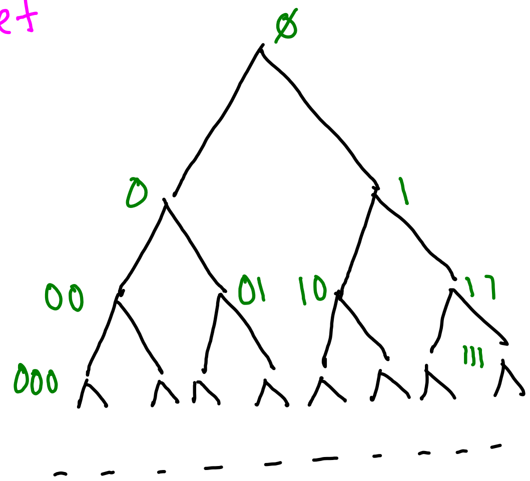
$\boxed{\text{More on self-similarity}}$

$X = \{x_1, \dots, x_d\}$, $d \geq 2$ - alphabet

$T_d = T(X)$ - d-regular rooted tree

$V(T_d) = X^*$ - set of all words over X

set of vertices



$\partial T = X^{\mathbb{N}}$ - Boundary

Tychonoff topology

$Aut(T_d)$ - group of automorphisms

permutational wreath product

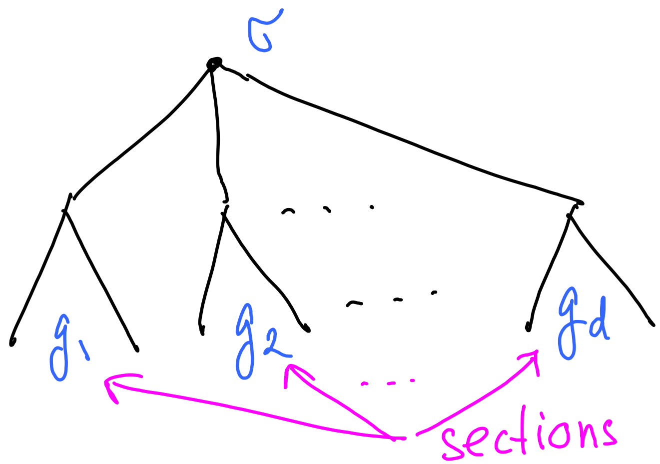
$Aut(T_d) \wr_X S_d$

$//$

$$\Psi : \mathrm{Aut}(T_d) \xrightarrow[\text{isomorphism}]{\simeq} \big(\underbrace{\mathrm{Aut}(T_d)\times\cdots\times\mathrm{Aut}(T_d)}_{d}\big) \rtimes \underbrace{S_d}_{\text{symmetric group}}$$

$$\mathrm{Aut}(T_d) \ni g \longleftrightarrow (g_1,\ldots,g_d)\sigma \;,\quad \sigma \in S_d$$

$g =$



$\mathrm{Aut}(T_d) > G$ – group acting faithfully on T_d

Def. G is said to be self-similar if $\forall g \in G$, sections $g_i \in G$, $i = 1, \dots, d$ after identification of T_d with subtrees with the root at first level.

$\Leftrightarrow$ $\forall g \in G$, $x \in X$ $\exists h \in G$, $y \in X$ s.t

$$\boxed{g(xw) = y\,h(w)} \quad \forall w \in X^*$$

h is section of g at vertex x, $y = \sigma(x)$.

Finitely generated self-similar groups are defined by (wreath) recursions

Examples: (i) $\mathcal{G} = \langle a, b, c, d \rangle$

$$a = (1,1)\sigma \qquad\qquad a = \sigma$$
$$b = (a,c)\cdot 1 \qquad\qquad b = (a,c)$$
$$c = (a,d)\cdot 1 \quad \Longleftrightarrow \quad c = (a,d)$$
$$d = (1,b)\cdot 1 \qquad\qquad d = (1,b)$$

(ii) $\mathcal{Y}_1 = \langle a, b \rangle$

cycle

$1, \sigma \in S_4$

$\sigma^4 = 1$

$a = (1,1,1,1)\sigma$ $\quad\Longleftrightarrow\quad$ $a = \sigma$

$b = (a, 1, a, b) \cdot 1$ $\qquad$ $b = (a, 1, a, b)$

(iii) $d \geqslant 3$, $\quad a = \sigma$, $\sigma^d = 1$

cycle

$$b = (a^{\varepsilon_1}, a^{\varepsilon_2}, \dots, a^{\varepsilon_{d-1}}, b), \quad 0 \leq \varepsilon_i \leq d-1$$

$G = \langle a, b \rangle$ - self-similar group for any choice of ε_j.

if $b = (a, a^{-1}, 1, \dots, 1, b)$ and $d = p \geq 3$ prime

get Gupta-Sidki p-groups.

Th. Let $d = p \geq 3$ be prime. Then $G = \langle a, b \rangle$ is infinite (torsion) p-group $\iff \sum_{j=1}^{p-1} \varepsilon_j \equiv 0 \pmod p$ and $(\varepsilon_1, \dots, \varepsilon_{p-1}) \neq (0, \dots, 0)$.

Growth (of Gupta-Sidki groups) is unknown (exponential or intermediate).

Notation: $St_G(n)$ - stabilizer of n-th level.

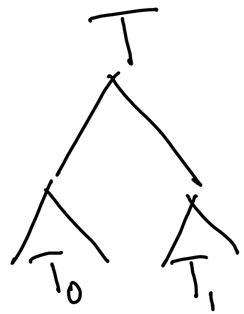
Flavor of proof.

$G = \mathfrak{G} = \langle a, b, c, d \rangle$

$\mathfrak{G} \underset{2}{\triangleright} H = St_{\mathfrak{G}}(1)$

$H = \langle b, c, d, aba, aca, ada \rangle$

$\psi : H \longrightarrow G \times G$ – *embedding*



$H \ni g \longrightarrow (g_0, g_1)$ $\qquad g_i = g|_{T_i}$ – restrictions (= sections)

$i = 0, 1$

$$\psi : \begin{cases} b \to (a, c) & aba \to (c, a) \\ c \to (a, d) & aca \to (d, a) \\ d \to (1, b) & ada \to (b, 1) \end{cases}$$

$\pi_i : G \times G \longrightarrow G$ projections

$i = 0, 1$

$$\begin{tikzcd}
G \arrow[d, dash, "2"'] & G\times G \arrow[d, dash] & \\
H \arrow[r, "\psi"] & \psi(H) \arrow[r, two heads, "\pi_0", "\text{surjection}"'] & G
\end{tikzcd}$$

$\Rightarrow$ G is <u>infinite</u> !

$$\psi((ad)^2) = \psi(\underbrace{ada}\underbrace{d}) = (b,1)(1,b) = (b,b)$$

$$\Rightarrow \psi((ad)^4) = (b^2, b^2) = (1,1) \Rightarrow (ad)^4 = 1$$

$$\psi((ac)^2) = \psi(\underbrace{aca}\underbrace{c}) = (d,a)(a,d) = (da,ad)$$

$$\Rightarrow \psi((ac)^8) = 1$$

Similarly $(ab)^{16}=1$, $(adacac)^4=1$, et.c

$\mathcal{G}_1$ is ~~infinite~~ f.g. infinite torsion 2-group (Burnside type group).

presentation by generators and relations

$$G=\langle a,b,c,d \mid 1=a^2=b^2=c^2=d^2=bcd=\tau^K((ad)^4)$$
$$=\tau^K((adacac)^4),\ K=0,1,2,\dots\rangle$$

$$\tau:\begin{cases} a\to aca \\ b\to d \\ c\to b \\ d\to c \end{cases}$$

— Lysenok's substitution (will play remarkable role later)