Groups of intermediate growth and aperiodic order.

Lecture 2.

Remind: groups $\mathcal{G}, \mathcal{G}_1, \dots,$ self-similar groups

$\mathcal{G} = \langle a, b, c, d \rangle \hookrightarrow T_2$ – Binary tree

group of intermediate growth

$$\left.\begin{array}{ll} a = \sigma & c = (a, d) \\ b = (a, c) & d = (1, b) \end{array}\right\} \text{– recursive relations}$$

(wreath recursion)

Getting lower bound.

Prop. $\mathcal{G}$ is commensurable to $\mathcal{G}^2$ $= \mathcal{G} \times \mathcal{G}$ direct product

Cor. $\boxed{\gamma_{\mathcal{G}}(n) \sim \gamma_{\mathcal{G}}^{2}(n)}$ $\Rightarrow \exists \alpha > 0$ s.t. $\boxed{\gamma_{\mathcal{G}}(n) \succ e^{n^{\alpha}}}$

$\Rightarrow$ superpolynomial growth.

Proof

$$\begin{tikzcd} \mathcal{G} \arrow[d, dash, "2"'] & & \mathcal{G} \times \mathcal{G} \arrow[d, dash] \arrow[dr, dash, "\leq 64"] & \\ \mathrm{St}_{\mathcal{G}}(1) = H \arrow[rr, "\psi"] & & \psi(H) > & B \times B \end{tikzcd}$$

$B = \langle b \rangle^{\mathcal{G}}$ ← normal closure

$\psi(d) = (1, b)$

$\psi(ada) = (b, 1)$

$\pi_i : \Psi(H) \twoheadrightarrow \mathcal{G}$ Self-replicating property.

projections $\quad i=0,1$ surjection (epimorphism)

$\Rightarrow \forall g \in \mathcal{G} \quad \exists h \in H$ s.t. $\Psi(h) = (f, g)$.

$\exists f \in \mathcal{G}$

$$d^h \xrightarrow{\Psi} (f^{-1}, g^{-1})(1, b)(f, g) = (1^f, b^g) = (1, b^g)$$

$\quad \shortparallel$

$h^{-1} d h$ - conjugation $\quad \Rightarrow \Psi(H) \geq 1 \times B$

Similar $\quad \Psi(H) \geq B \times 1 \quad \Rightarrow \Psi(H) \geq B \times B.$

$\mathcal{G}/B = \langle \bar{a}, \bar{d} \mid \dots \rangle$ - dihedral group of order ≤ 8

as $\bar{b} = 1$, $\bar{c} = \bar{d}$ as $bcd = 1$

in fact $\quad \gamma_{\omega}(n) > e^{n^{\frac{1}{2}+0.04}}$

Getting an upper Bound (idea).

(i) Contracting property.

Def. A self-similar f.g. group $G \curvearrowright T_d$ is contracting if $\exists\ \lambda < 1,\ C$ s.t. $\forall g \in G$

$$g = (g_1, \dots, g_d)\sigma \leftarrow \text{first level recursion}$$

$$\boxed{|g_i| < \lambda |g| + C} \quad i = 1, \dots, d$$

$\Rightarrow \boxed{|g_i| < |g|}$ if $|g| > \frac{C}{1-\lambda}$

sections g_i are shorter than g.! This allows to prove many things: torsion property, ...

Example. $\mathcal{G}$, $\mathcal{G}_1$, ... are contracting with $\lambda = \frac{1}{2}$, $C = 1$.

Example. $\mathcal{B} = \langle a, b \rangle \hookrightarrow T_2$

$a = (1, b)$
$b = (1, a)\sigma$

recursive relations

Basilica (= IMG(z^2-1))

is contracting with parameters $\lambda = \frac{2}{3}$, $C = 1$

Example. $\tilde{\mathcal{J}} \ni g = abacadabadac$

length $= 12$

$|g| = 12$

$$= (c,a)(a,d)(b,1)(a,c)(b,1)(a,d)$$

$$= (cababa,\ ad\cdot 1\cdot c1d) = (\underbrace{cababa}_{g_1},\ \underbrace{ac}_{g_2})$$

$$|g_1| = 6 = \frac{|g|}{2}$$

$$|g_2| = 2\ ,$$

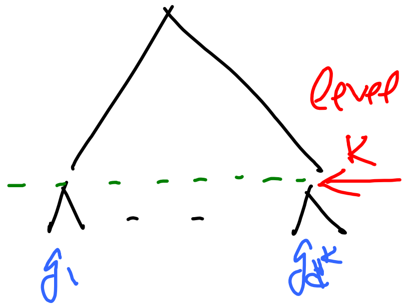
g_1, g_2 more than twice shorter than g

More general version of the definition of contracting:

Def. G is contracting with parameters

$\lambda < 1$, C, $k \in \mathbb{N}$ if $\forall g \in G$

$$g = (g_1, \dots, g_{d^k})\sigma$$



$(g_1, \dots, g_{d^k})$ sections of g at vertices of level k, $\sigma \in S_{d^k}$ – symmetric group

$$\boxed{|g_i| < \lambda |g| + C, \quad i = 1, \dots, d^k}$$

Th. (Nekrashevych). Self-similar contracting group does not contain a free subgroup.

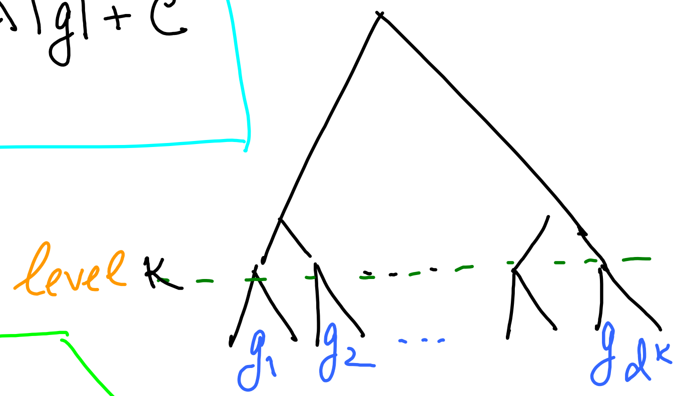
Question. Are self-similar contracting groups amenable? (confirmed in many cases but still open).

Def. A f. g. self-similar group G is strongly contracting if $\exists\ \lambda < 1$, C and $k \in \mathbb{N}$ s.t.

$\forall g \in G$, $g = (g_1, \dots, g_{d^k})\sigma$ where $g_1, \dots, g_{d^k}$ are sections of g at vertices of k-th level, and $\sigma \in S_{d^k}$ the inequality also called sum contracting property

$$\boxed{\sum_{i=1}^{d^k} |g_i| < \lambda |g| + C}$$

holds.



Th. For each strongly contracting group G with parameters λ, C, k there is $\beta = \beta(\lambda, k) < 1$ s.t.

$$\boxed{\gamma_G(n) < e^{n^\beta}}$$

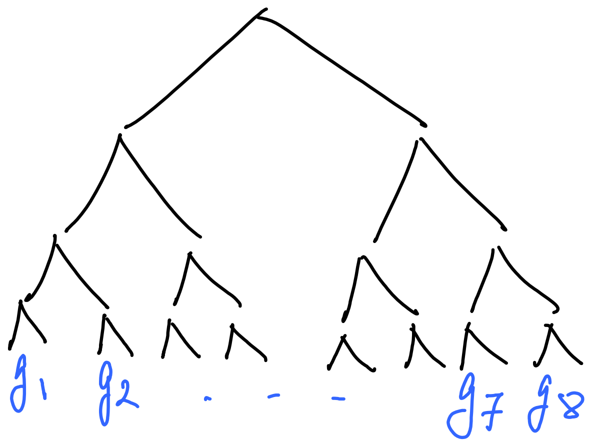
$\Rightarrow$ G has subexponential growth.

Example. $\mathcal{G} = \langle a, b, c, d \rangle$ has strong contracting property with parameters $\lambda = \frac{3}{4}$, $C = 10$, $K = 3$.

$$\mathcal{G} \ni g = (g_1, \dots, g_8)\sigma$$

$$\Downarrow$$

$$\sum_{i=1}^{8} |g_i| < \frac{3}{4}|g| + 10$$



$\Rightarrow$ $\mathcal{G}$ has intermediate growth as

$$e^{n^{\alpha}} \prec \gamma_{\omega}(n) \prec e^{n^{\beta}}$$

where $0 < \alpha < \beta < 1$. in fact !

$$e^{n^{\frac{1}{2}+0.04}} \preccurlyeq \gamma_{\omega}(n) \preccurlyeq e^{n^{0.7574}}$$

(precise growth is unknown).

Def. G satisfies the anti-contracting property with parameters $\mu, C, \kappa \in \mathbb{N}$ if $\forall g \in G$

$$g = (g_1, \dots, g_{d^k})\sigma$$

$$\boxed{|g| \leq \mu \sum_{i=1}^{d^k} |g_i| + C} \qquad (*)$$

Th. if G satisfies $(*)$ then $\exists\ \alpha = \alpha(k,\mu) > 0$

s.t.

$$\boxed{\gamma_G(n) \gtrsim e^{n^{\alpha}}}$$

Example. $\mathfrak{G}$ satisfies $(*)$ with parameters

$\mu = 2$, $k = 1$, $C = 1$. if $g = (g_1, g_2)\sigma$ then

$$|g| \leq 2(|g_1| + |g_2|) + 1$$

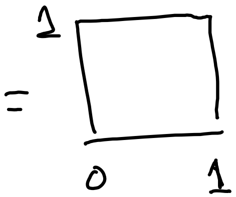
and $\alpha(2,1) = \frac{1}{2}$.

The upper bounds on growth are more important than lower. as many group properties imply that the group is not virtually nilpotent.

For instance: to be torsion, to be simple, not to be residually finite, ...

Groups of intermediate growth can have all these (and many other) properties.

Dynamical systems associated with groups acting on rooted trees.

$\mathcal{Y} \curvearrowright [0,1]$ $\qquad$ $\mathcal{Y}_1 \curvearrowright [0,1]^2 =$ 

square

actions preserving Lebesgue measure

dynamical system

$\mathcal{D} = (G, X, \mu)$

group $\quad$ space $\quad$ measure

μ $\begin{cases} \text{finite (probability)} \\ \text{infinite} \end{cases}$

μ $\begin{cases} \text{invariant} \\ \text{quasi-invariant} \end{cases}$

1970th — dynamical system with non-commutative time (schools of Sinai, Anosov, ...)

2010th — measured group theory

Examples: (i) $(G, \mathrm{Sub}(G), \mu)$

G acts by conjugation

space of subgroups

IRS (invariant random subgroup)

(ii) $(\mathrm{Aut}\, G, \mathrm{Sub}(G), \nu)$

CRS (characteristic random subgroups)

group of automorphisms of G acts on Sub(G):

$G > H \xrightarrow{\varphi} H^{\varphi}, \quad \varphi \in \mathrm{Aut}\, G.$

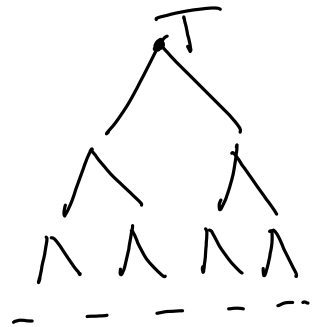
$G \curvearrowright T = T_d$

$G \curvearrowright \partial T$ - boundary

= space of geodesic paths joining the root with infinity

= space of sequences of alphabet of cardinality d

$(G, \partial T)$ - topological dynamical system



$\partial T = \{0, 1, \dots, d-1\}^{\mathbb{N}}$

IS

Cantor set

$p = \{\frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}\}$.

uniform distribution on the alphabet

$\mu = \bigtimes_{\mathbb{N}} P$ — product measure (= uniform

Bernoulli measure on ∂T)

μ is invariant w.r. $\mathrm{Aut}(T)$

$(G, \partial T, \mu) \underset{\text{isomorphism}}{\simeq} (G, [0,1], \text{Lebesgue})$

Orbital graphs (graphs of action)

G – finitely generated, $S = \{s_1, .., s_m\}$ – system of generators.

$G \curvearrowright X$ – topological or measure space

$X \ni x \rightsquigarrow \Gamma_x = (V_x, E_x)$

vertices edges

$V_x = O_G(x) = \{gx \mid g \in G\}$

orbit

$$E_x = \{(gx, sgx) \mid g \in G,\ s \in S\}$$

label $\longrightarrow$ s

$$gx \xrightarrow{\ s\ } sgx$$

Get a family $\{\Gamma_x : x \in X\}$ of $2|S|$ - regular graphs.

$$\Gamma_x \simeq \Gamma(G, H, S) \quad \text{where } H = St_G(x) = \{g \in G : gx = x\}$$

Schreier graph

stabilizer

$$\Gamma(G, H, S) = (V, E)$$

$$V = \{ gH \mid g \in G \}$$

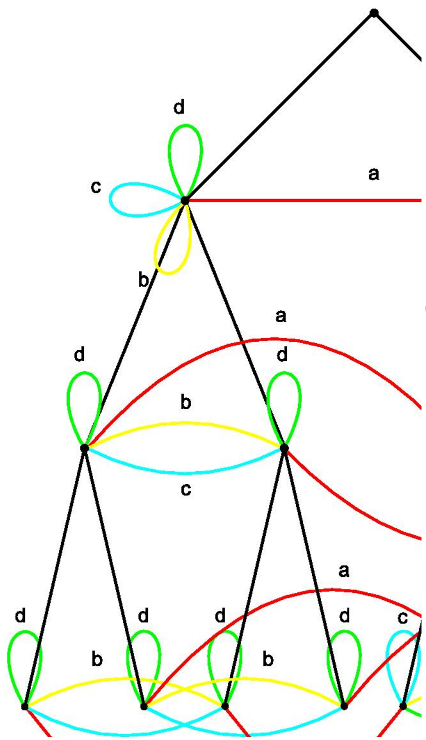
left cosets

$$E = \{ (gH, sgH) \mid g \in G,\ s \in S \}$$

$$gH \xrightarrow{s} sgH$$

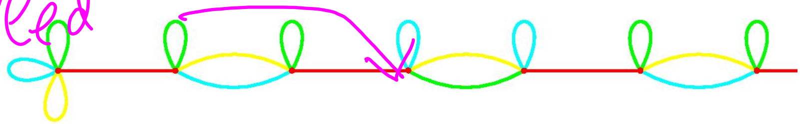
Example.

$\mathcal{G} = \langle a, b, c, d \rangle$



orbital graphs of action on levels 1, 2, 3 unlabelled

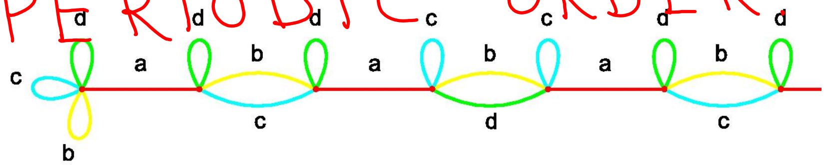
labelled



orbital graph of point $1^\infty \in \partial T$

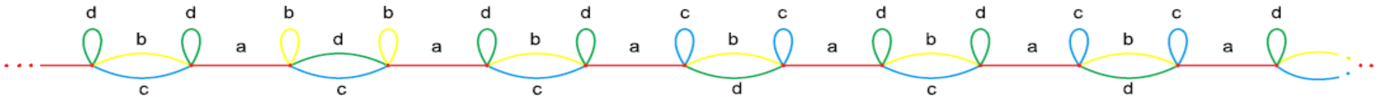
A linear type 1-ended Schreier graph

APERIODIC ORDER!



next page: – two ended linear graphs

two ends



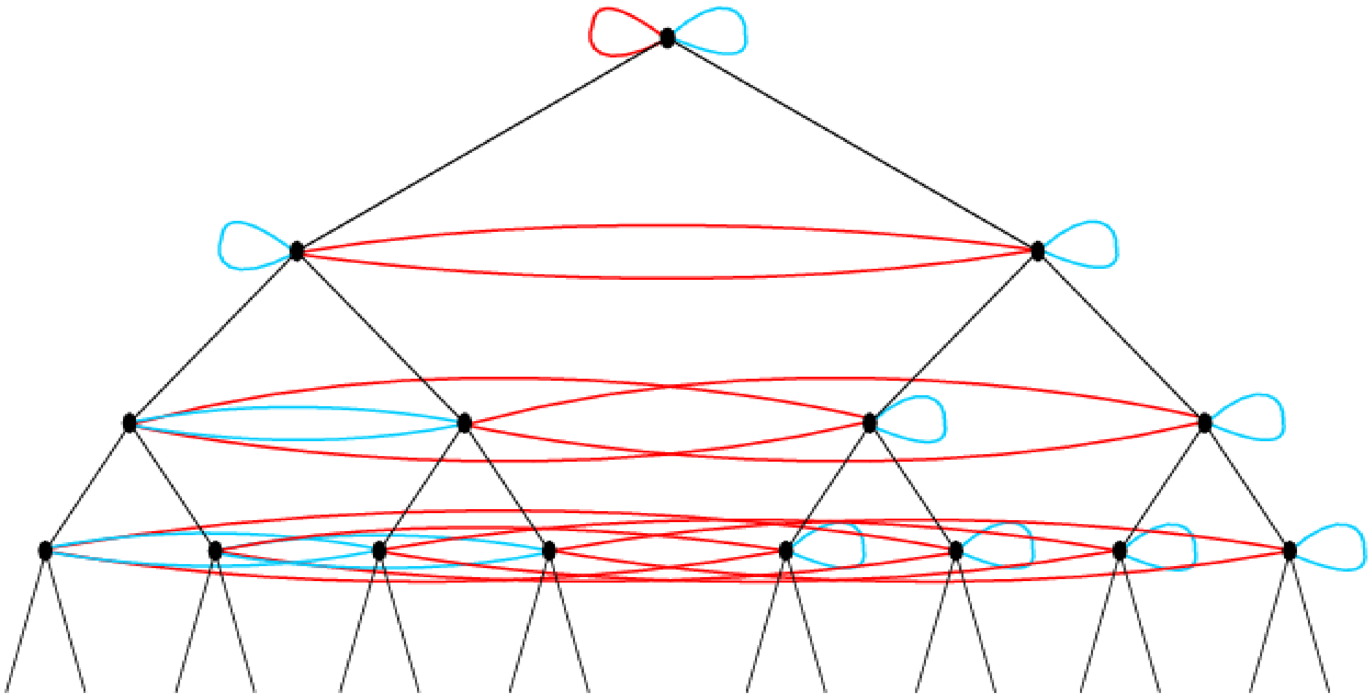
Γ_x is 1-ended $\Longleftrightarrow$ $x \in O(1^\infty)$

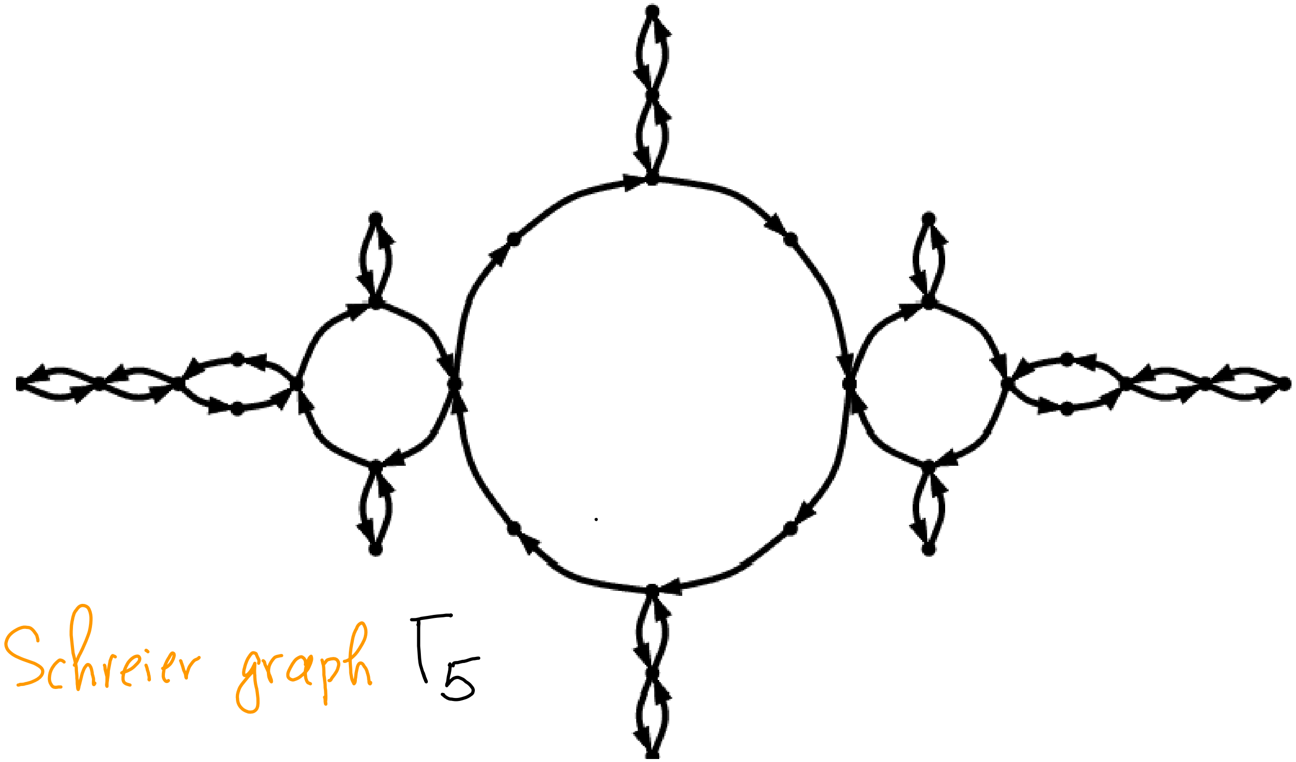
$\Rightarrow$ typical case when Γ_x is two ended

More examples:

Basilica $\quad B = \langle a, b \rangle \quad a = (1, b)$

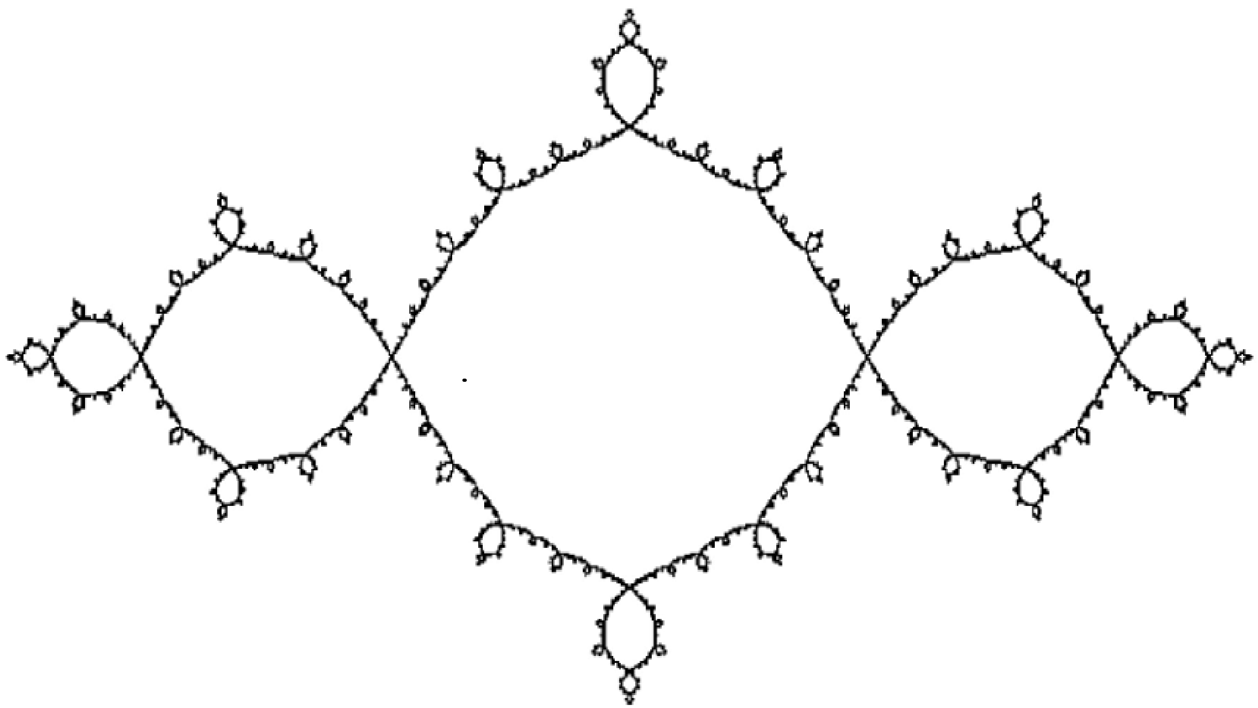
$b = (1, a)\sigma$





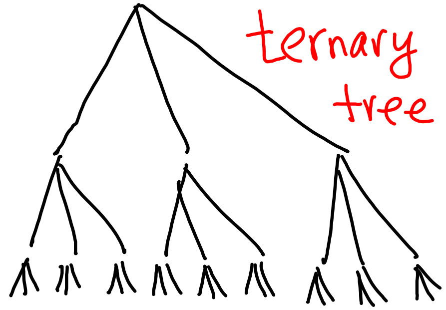
Schreier graph Γ_5

$\Gamma_n \xrightarrow{n\to\infty}$ Julia set of z^2-1



Hanoi Tower group $H^{(3)}$ (three pegs)

T_3



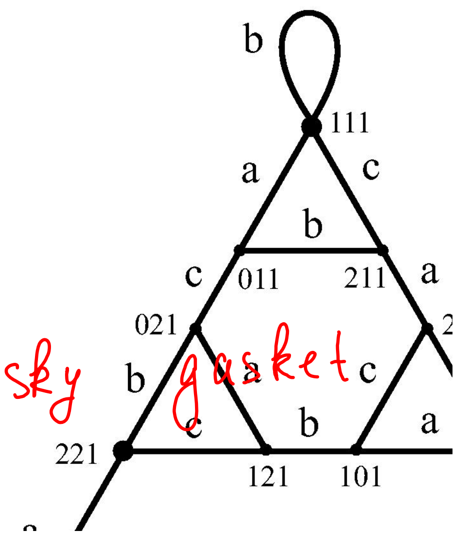
$$H^3 = \langle a, b, c \rangle$$

$$a = (1, a, a)\sigma_0$$
$$b = (b, 1, b)\sigma_1$$
$$c = (c, c, 1)\sigma_2$$

$$\left.\begin{array}{l} \sigma_0 = (1\ 2) \\ \sigma_1 = (0\ 2) \\ \sigma_2 = (0, 1) \end{array}\right\} \in S_3 \curvearrowright \{0, 1, 2\}$$

symmetric group

Schreier graph Γ_3 of $H^{(3)}$



$\Gamma_n \xrightarrow{n \to \infty}$ Sierpinsky gasket