Groups of intermediate growth and aperiodic order.

Lecture 2.

Remind: groups $\mathcal{Y}, \mathcal{Y}_1, \ldots$, self-similar groups

$\mathcal{Y} = \langle a, b, c, d \rangle \rightarrow T_2$ – Binary tree

Group of intermediate growth

$a = \sigma$  \quad  $c = (a, d)$

$b = (a, c)$  \quad  $d = (1, b)$ – recursive relations

(wreath recursion)
Getting lower bound.

Prop. \( J \) is commensurable to \( J^2 \)

Cor. \( \gamma_y(n) \sim \delta_y^2(n) \Rightarrow \exists \alpha > 0 \text{ s.t.} \delta_y(n) \geq e^{n^2} \Rightarrow \text{super polynomial growth.} \)

Proof

\[
\begin{align*}
\gamma & | \gamma \times \gamma \\
2 & | \psi \\
S_{2y}(1) = H & \rightarrow \psi(H) > B \times B \\
\psi(d) = (1,8) & \\
\psi(ada) = (8,1) & \\
B = \langle 8 \rangle & \uparrow \text{normal closure}
\end{align*}
\]
\[ \mathcal{Y}_i : \Psi(H) \to \mathcal{Y} \]

\[ \text{surjection (epimorphism)} \]

\[ \text{projections } i = 0, 1 \]

\[ \Rightarrow \forall g \in \mathcal{Y} \exists h \in H \text{ s.t. } \Psi(h) = (f, g). \]

\[ \exists f \in \mathcal{Y} \]

\[ d^h \Psi (f^{-1} g^{-1}) (1, b) (f, g) = (1, b) (f, g) = (1, b) \]

\[ h^{-1} d^h - \text{ conjugation} \Rightarrow \Psi(H) \geq 1 \times B \]

Similar \[ \Psi(H) \geq B \times 1 \Rightarrow \Psi(H) \geq B \times B. \]

\[ \mathcal{Y} / B = \langle \bar{a}, \bar{d} | \ldots \rangle \]

\[ \text{- dihedral group of order } \leq 8 \]

as \[ \bar{b} = 1, \bar{c} = \bar{a} \]

as \[ b c d = 1 \]
In fact $\chi_{wy}(n) \geq e^{n^{1/2} + 0.04}$

Getting an upper bound (idea).

(i) Contracting property.

Def. A self-similar group $G \to T_d$ is contracting if $\exists \lambda < 1, C$ s.t. $\forall g \in G$

$$g = (g_1, \ldots, g_d) \overset{\text{first level}}{\rightarrow}$$

$$|g_i| < \lambda |g| + C \quad i = 1, \ldots, d$$
\[ \Rightarrow 1g_i \leq 1g_1 \quad \text{if} \quad 1g_1 > \frac{C}{1-\lambda} \]

Sections \( g_i \) are shorter than \( g_1 \). This allows to prove many things: torsion property, ...

Example. \( Y, Y_1, \ldots \) are contracting with \( \lambda = \frac{1}{2}, C = 1 \).

Example. \( B = \langle a, b \rangle \rightarrow T_2 \quad | \quad a = (1, b) \quad b = (1, a) \)

Basilica \( (= \text{IMG}(z^2 - 1)) \) is contracting with parameters \( \lambda = \frac{2}{3}, C = 1 \).
Example. \( g = \text{abac adabadac} \)

\[ \text{Length} = 12 \]

\[ |g| = 12 \]

\[ (c, a)(a, d)(c, b)(a, c)(b, c)(a, d) \]

\[ (\text{cababa, ad} \cdot \text{ca} \cdot \text{id}) = (\text{cababa, ac}) \]

\[ |g_1| = 6 = \frac{|g|}{2} \]

\[ |g_2| = 2 \]

\( g_1, g_2 \text{ more than twice shorter than } g \)
More general version of the definition of contracting:

**Def.** $G$ is contracting with parameters $\lambda < 1$, $C$, $k \in \mathbb{N}$ if $\forall g \in G$

$$g = (g_1, \ldots, g_d, k)$$

$(g_1, \ldots, g_d, k)$ sections of $g$ at vertices of level $k$, $\sigma \in S_d^k$ - symmetric group

$$|g_i| < \lambda |g| + C, \quad i = 1, \ldots, d$$
Th. (Nekrashevych). Self-similar contracting group does not contain a free subgroup.

Question. Are self-similar contracting groups amenable? (Confirmed in many cases but still open).

Def. A f. g. self-similar group is strongly contracting if \( \exists \lambda < 1, C \) and \( k \in \mathbb{N} \) s.t.

\[ \forall g \in \mathcal{G} \quad g = (g_1, \ldots, g_{d_k})^\mathcal{G} \quad \text{where} \]

\( g_1, \ldots, g_{d_k} \) are sections of \( g \) at vertices of \( k \)-th level, and \( \mathcal{G} \in \mathcal{S}_{d_k} \) the inequality
Theorem. For each strongly contracting group $G$ with parameters $\lambda, \zeta \in \mathbb{K}$, there is $\beta = \beta(\lambda, k) < 1$ s.t. 

$$\gamma_{G_n} < C n^\beta$$

$\Rightarrow$ $G$ has subexponential growth.
Example. \( \mathcal{Y} = \langle a, b, c, d \rangle \) has strong contracting property with parameters \( \lambda = \frac{3}{4}, C = 10, k = 3. \)

\[ \mathcal{Y} \ni g = (g_1, \ldots, g_8) \]

\[ \sum_{i=1}^{8} |g_i| < \frac{3}{4} |g_1| + 10 \]

\[ \Rightarrow \mathcal{Y} \text{ has intermediate growth as} \]
\[ e^\alpha < \delta_y(n) < e^{n\beta} \]

where \( 0 < \alpha < \beta < 1 \). In fact,

\[ n^{0.7574} < \delta_y(n) < n^{\frac{1}{2} + 0.04} \]

( precise growth is unknown ).

**Def.** \( G \) satisfies the anti-contracting property with parameters \( \mu, C, \kappa \in \mathbb{N} \) if \( \forall g \in G \)

\[ g = (g_1, \ldots, g_{2^\kappa}) \]
\[ |g| \leq \mu \sum_{i=1}^{d^k} |g_i| + c \]  \hspace{1cm} (\star)

**Theorem.** If \( G \) satisfies (\( \star \)) then \( \exists \alpha = \alpha(k, \mu) > 0 \) s.t.

\[ \gamma(G(n)) \geq e^{\alpha n} \]

**Example.** \( Y \) satisfies (\( \star \)) with parameters \( \mu = 2, \ k = 1, \ c = 1 \). If \( \gamma = (g_1, g_2)^c \) then

\[ |g| \leq 2(1g_1 + 1g_2) + 1 \]

and \( \alpha(2, 1) = \frac{1}{2} \).
The upper bounds on growth are more important than lower, as many group properties imply that the group is not virtually nilpotent.

For instance: to be torsion, to be simple, not to be residually finite, ...

Groups of intermediate growth can have all these (and many other) properties.
Dynamical systems associated with groups acting on rooted trees.

\[ \mathcal{Y} \subset [0,1] \]
\[ \mathcal{Y}_1 \subset [0,1] \]

Dynamical system

\[ \mathcal{D} = (G, X, \mu) \]

Group \ Space \ Measure

Actions preserving Lebesgue measure

Square

Finite (probability)

Invariant quasi-invariant
1970th - dynamical system with non-commutative time (schools of Sinai, Anosov, ...)

2010th - measured group theory

Examples:
(i) \((G, \text{Sub}(G), \mu)\)

\(G\) acts by conjugation on the space of subgroups \(\text{Sub}(G)\).

(ii) \((\text{Aut } G, \text{Sub}(G), \mathcal{V})\)

Group of automorphisms of \(G\) acts on \(\text{Sub}(G)\):
\[G \xrightarrow{\varphi} H^\varphi, \quad \varphi \in \text{Aut } G.\]
$G \rightarrow T = T_d$

$G \rightarrow \partial T$ - boundary

= space of geodesic paths joining the root with infinity

= space of sequences of alphabet of cardinality $d$

$(G, \partial T)$ - topological dynamical system

\[
\mathcal{T} = \{0, 1, \ldots, d-1\}^\mathbb{Z}
\]

IS Cantor set

\[
p = \{\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d}\}
\]

uniform distribution on the alphabet
$$\mu = \bigotimes_{N} P \quad \text{— product measure (}= \text{uniform Bernoulli measure on } \partial T)$$

$$\mu \text{ is invariant w.r. } \text{Aut}(T)$$

$$(G, \partial T, \mu) \simeq (G, [0,1], \text{Lebesgue})$$

isomorphism
Orbital graphs (graphs of action)

$G$ - finitely generated, $S = \{s_1, \ldots, s_m\}$ - system of generators.

$G \subseteq X$ — topological or measure space

$x \mapsto \Gamma_x = (V_x, E_x)$

$V_x = O_G(x) = \{g \cdot x \mid g \in G\}$
\[ E_x = \{ (gx, sgx) \mid g \in G, s \in S \} \]

Get a family \( \{ \Gamma_x : x \in X \} \) of 21SI-regular graphs.

\[ \Gamma_x \cong \Gamma(G, H, S) \]

where \( H = S^G_x(x) \)

\( \uparrow \) Schreier graph
\[ \Gamma(G, H, S) = (V, E) \]

\[ V = \{ gh \mid g \in G \} \]

*left cosets*

\[ E = \{ (gh, sgH) \mid g \in G, s \in S \} \]

\[ gH \quad \xrightarrow{s} \quad sgH \]
Example.

\[ y = \langle a, b, c, d \rangle \]

Orbital graphs of action on levels 1, 2, 3.
A linear type 1-ended Schreier graph

A PERIODIC ORDER!

next page: - two ended linear graphs
two ends

$\Gamma_x$ is 1-ended $\iff x \in O(1^\infty)$

$\Rightarrow$ typical case when $\Gamma_x$ is two ended

More examples:
Basilica \( B = \langle a, b \rangle \)  
\[ a = (1, b) \]  
\[ b = (1, a)^{\overline{5}} \]
Schreier graph $\Gamma_5$
$\Gamma \underset{n \to \infty}{\longrightarrow} \text{Julia set of } \mathbb{C}_2 - 1$
Hanoi Tower group $H^3$ (three pegs)

$H^3 = \langle a, b, c \rangle$

$a = (1, a, a) \sigma_0$

$b = (b, 1, b) \sigma_1$

$c = (c, c, 1) \sigma_2$

$\sigma_0 = (1, 2)$

$\sigma_1 = (0, 2)$

$\sigma_2 = (0, 1)$

$\epsilon \in S_3 \to \{0, 1, 2\}$

ternary tree

symmetric group
Schreier graph $\Gamma_3$ of $H^{(3)}$