Lecture 3.

Automaton presentation of self-similar groups.

Automata → acceptors

acceptors — formal languages

acceptors — hyperbolic group (Gromov)

acceptors — automatic groups (groups with automatic structure)

acceptors — Automata groups

Automata ↓ transducers (Mealy or Moore type)

transducers → automatically generated sequences

transducers → Automata groups (self-similar groups)

$\mathcal{A} = \langle X, Q, \varphi, \psi \rangle$

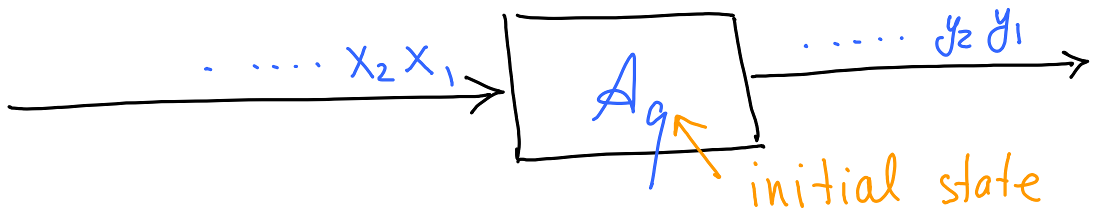
$\mathcal{A}_q$ – automaton with initial state q
(initial automaton)

X – alphabet

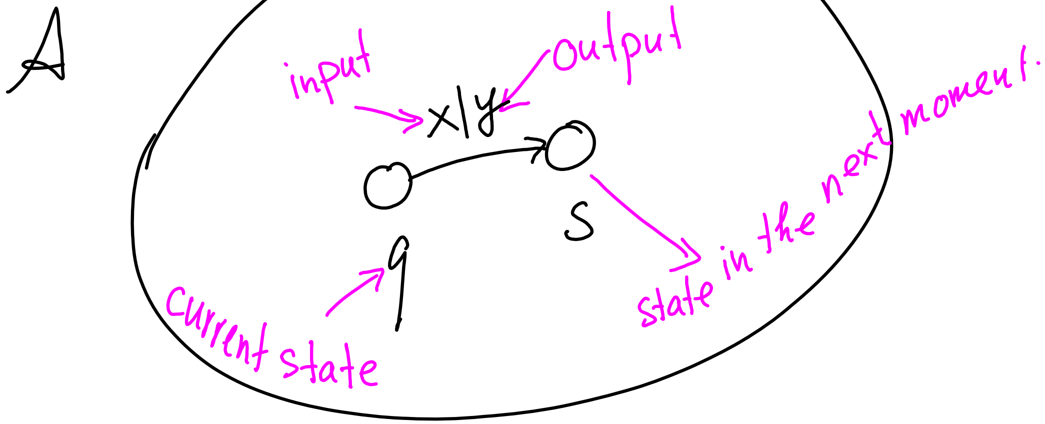
Q – set of states

$\varphi : X \times Q \longrightarrow Q$ – transition function

$\psi : X \times Q \longrightarrow X$ – exit (output) function



$$y = \psi(x, q)$$
$$s = \varphi(x, q)$$



x_1	x_2	x_3			- - - -	x_n	x_{n+1}	. . .
y_1	y_2	y_3			- - -	y_n	y_{n+1}	. . .
q_1	q_2	q_3			- - - -	q_n	q_{n+1}	

$$y_{n+1} = \psi(x_n, q_n), \qquad q_{n+1} = \varphi(x_n, q_n)$$

initial automaton determines maps

$$X^n \longrightarrow X^n \qquad n = 1, 2, \ldots$$

$$X^{\mathbb{N}} \longrightarrow X^{\mathbb{N}}$$ – space of infinite sequences

if $Q = \{q_1, \ldots, q_m\}$ then

$S(A) = \langle A_{q_1}, \ldots, A_{q_m} \rangle$ – semigroup generated by A. it acts faithfully on $X^{\mathbb{N}}$. Operation – composition of maps.

$\mathcal{A}$ is invertible if $\forall q \in Q$

$$\sigma_q : X \to X, \qquad \sigma_q(x) = \Psi(x, q)$$

is a permutation of the alphabet:

$$\sigma_q \in S(X) \text{ — symmetric group}$$

In this case $\mathcal{A}_q$ is invertible.

$$G(\mathcal{A}) = \langle \mathcal{A}_{q_1}, \mathcal{A}_{q_1}^{-1}, \ldots, \mathcal{A}_{q_m}, \mathcal{A}_{q_m}^{-1} \rangle$$

group generated by automaton $\mathcal{A}$.

composition of maps $A_q \circ B_s$ corresponds to composition of automata $A_q * B_s$ (also automaton)

inverse of automaton A_q is automaton A_q^{-1}.

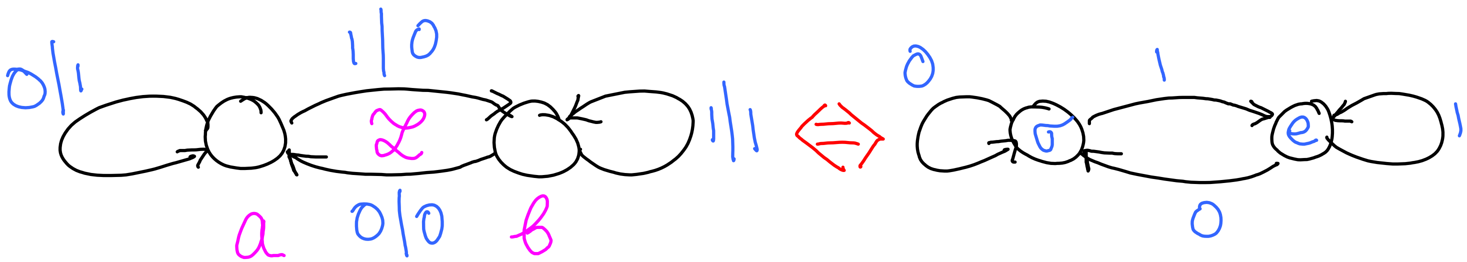
$\mathcal{F}A(X)$ – group of finite invertible initial automata over alphabet X.

$$\mathcal{F}A(X) < \mathrm{Aut}(T_X)$$

↑ rooted tree corresponding to X.

Lamplighter

$S_2 = \{e, \sigma\} \curvearrowright \{0, 1\}$



$$G(\mathcal{L}) \simeq \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} = \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}\Big) \rtimes \mathbb{Z}$$

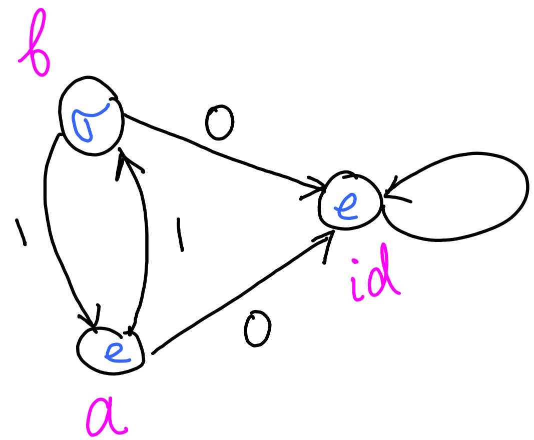
metabilian infinitely presented group of exponential growth.

(m, n) – automaton groups – groups generated by automata with m states over an alphabet of cardinality n.

6 groups of type $(2, 2)$

$\leqslant 115$ groups of type $(2,3)$

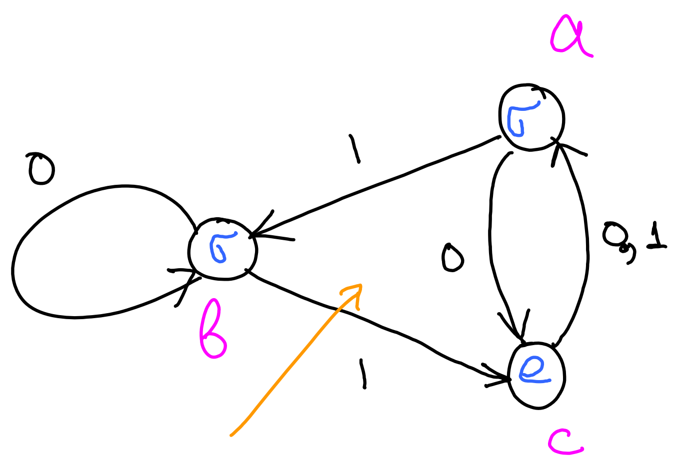
Basilica



$B \simeq IMG(z^2-1)$

iterated monodromy group

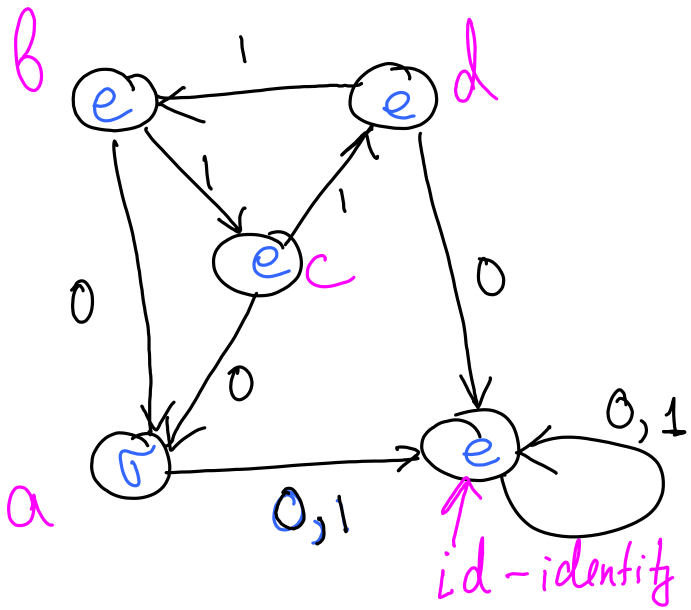
F_3 – free group



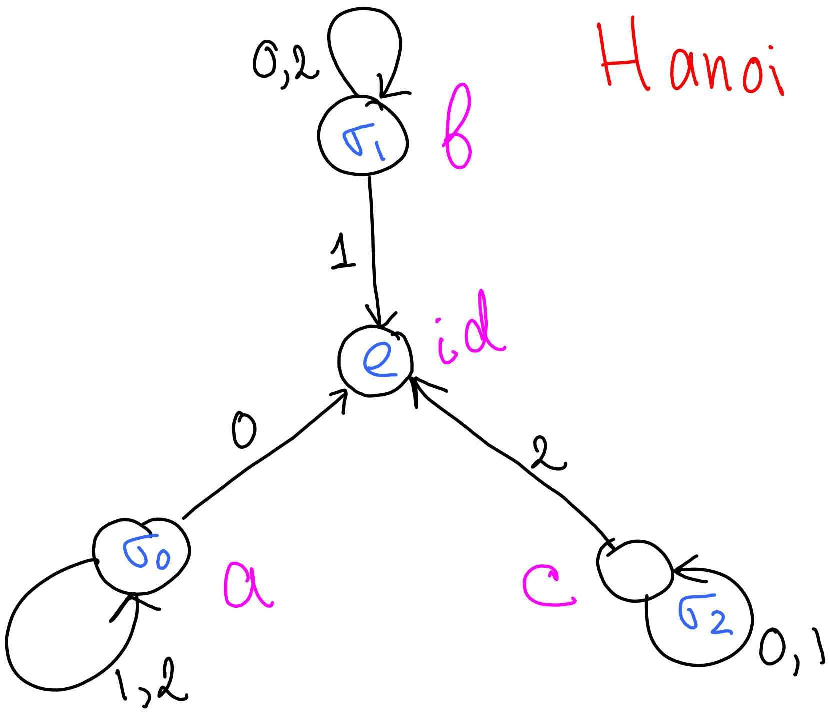
Aleshin automaton 1980

Very nontrivial proof that it generates F_3 given by Yaroslav & Mariya Vorobets

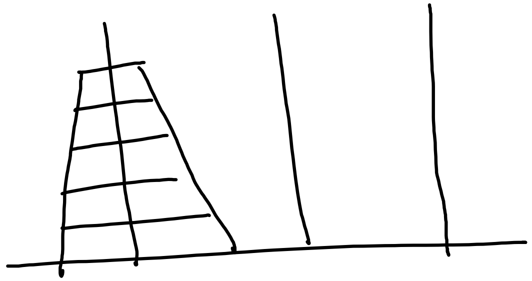
$\mathcal{G} = \langle a, b, c, d \rangle$

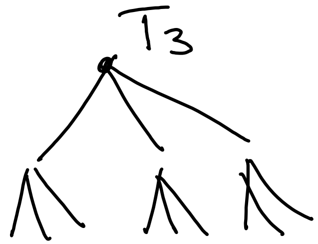


automaton presentation of $\mathcal{G}$



Hanoi $H^{(3)}$





ternary tree

Spectra of groups and graphs

Γ – finite or infinite d-regular graph

$$(Mf)(x) = \frac{1}{d} \sum_{y \sim x} f(y), \quad f \in \ell^2(\Gamma)$$

adjacent

Markov operator

$\Delta = I - M$ – discrete Laplacian

$Sp(M)$ – spectrum of Γ, $Sp(M) \subset [-1, 1]$

$$M = \int_{-1}^{1} \lambda \, dE(\lambda)$$ – spectral decomposition

$v \in V(\Gamma)$

$$\sigma_v(\lambda) = \langle E(\lambda) \delta_v, \delta_v \rangle$$ – spectral function

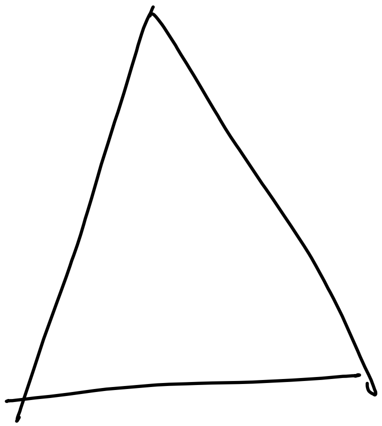
delta function

μ_v – spectral measure.

if Γ is transitive graph $\Rightarrow$ μ_v does not depend on v

We are interested in $sp(M)$, μ (spectrum, spectral measure). for Cayley graphs and for Schreier graphs.

Let $G \hookrightarrow T_d$



Γ_n

Γ_n – graph of action on n-th level

$|V(\Gamma_n)| = d^n$

Γ_n are connected $\Longleftrightarrow$ action is level transitive

$\Longleftrightarrow$ $(G \curvearrowright \partial T, \mu)$ is ergodic (invariant sets have measure 0 or 1).

uniform measure.

$$\pi : G \longrightarrow U(L^2(\partial T, \mu))$$

Koopman unitary representation

$$(\pi_g f)(x) = f(g^{-1}x), \quad f \in L^2(X, \mu)$$

Quasi-regular representation

$G > H$ – subgroup $\qquad G/H = \{gH \mid g \in G\}$ – coset set

$\lambda_{G/H}$ – quasi-regular repres. in $\ell^2(G/H)$

permutational representation as

$$G \curvearrowright G/H$$

$H = \{1\} \implies$ regular representation λ_G in $\ell^2(G)$

trivial subgroup

Hecke type operators

$\rho : G \longrightarrow U(H)$ – unitary representation

↑ Hilbert space

$m = \sum_{g \in G} a_g \, g \in \mathbb{C}[G]$, $\quad a_g \in \mathbb{C}$

↑ group algebra

$\rho(m) = \sum_g a_g \, \rho(g)$ – Hecke type operator

Examples:

G, S - finite system of generators

$$m = \frac{1}{2|S|} \sum_{s \in S} (s + s^{-1})$$

$\rho = \lambda_G$ - regular

$\rho(m)$ — Markov operator on Cayley graph

$$\Gamma = \Gamma(G, S)$$

$$m = \sum_{s \in S} a_s (s + s^{-1}), \quad a_s \in \mathbb{C}$$

$\rho(m)$ – "Markov" operator with weights a_s

if $\rho = \lambda_{G/H}$ then $\lambda_{G/H}(m)$ – Markov (or weighted Markov – Hecke type) operator on Schreier graph $\Gamma = \Gamma(G, H, S)$.

$G \curvearrowright X$

Family of Schreier graphs

$\{\Gamma_x \mid x \in X\}$, $V(\Gamma_x) = O_G(x)$ – orbit of x

Family of quasi-regular representations

ρ_x in $\ell^2(O_G(x))$, $x \in X$

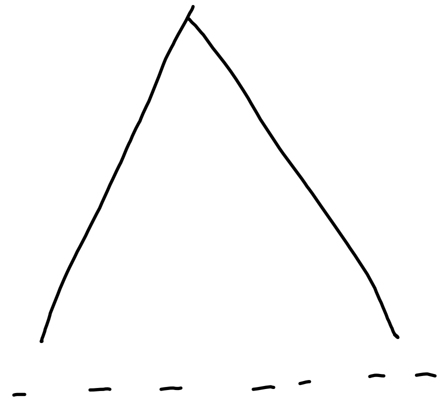
$m \in \mathbb{C}[G]$

family of operators $\rho_x(m)$, $x \in X$

We are interested in typical properties of these operators (spectra, spectral measures, ...)

Some results:

$G \quad G \curvearrowright T_d$ – d-regular tree



$\partial T = \{0, \dots, d-1\}^{\mathbb{N}}$

$\Downarrow$

$G \quad G \curvearrowright \partial(T_d)$

$(G, \partial(T_d), \mu)$

π – Koopman representation in $L^2(\partial T_d, \mu)$

π_n – permutational representation in $\ell^2(V_n)$

vertices of level n

ρ_x – permutational representation in $\ell^2(O_G(x))$

$$m = \sum_{s \in S} a_s (s + s^{-1}) \in \mathbb{C}[G$$

self-adjoint element.

Th. Bartholdi – Gri. 2000

a) $$\boxed{sp(\pi(m)) = \overline{\bigcup_{n \geq 0} sp(\rho_n(m))}}$$

and $sp(\rho_n(m)) \subset sp(\rho_{n+1}(m))$, $n = 1, 2, \dots$

(as Γ_{n+1} covers Γ_n)

b) $\forall\ x \in \partial T_n$

$$\boxed{sp(\rho_x(m)) \subset sp(\pi)}$$

c) if Γ_x is amenable then

$$\boxed{sp(\rho_x(m)) = sp(\pi).}$$

Remark. (i) π is a sum of finite-dimensional representations.

(ii) if G is amenable then Γ_x is amenable for $\forall x \in \partial T_d$.

(iii) in many cases ρ_x is infinite-dimensional and irreducible. So $\pi(m)$ and $\rho_x(m)$ are very different operators.

in particular this holds if G is Branch or even weakly branch.

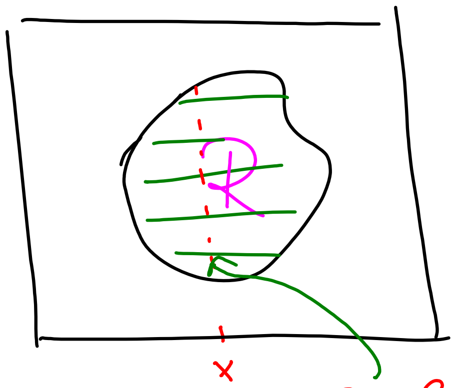
About Branch group (may be) later.

One more result in this direction.

(G, X, μ) $\quad \mu(X)=1$, $\quad \mu$ – probability, quasi-invariant!

$X \times X \supset R$ – orbit equivalence relation

$\nu = \mu \times$ counting measure



R_x - leave

$= O_G(x)$

$\mathcal{H} : G \longrightarrow U(L^2(R, \nu))$ - groupoid representation

$$(\mathcal{H}_g f)(x,y) = f(g^{-1}x, y)$$

Theorem. Dudko - Gri. 2015.

a) For an ergodic measure class preserving action of a countable group G on a probability space (X, μ) and any $m \in \mathbb{C}[G]$ one has

$$sp(\pi(m)) \supset sp(\rho_x(m)) = sp(\mathcal{H}(m))$$

for μ-almost all $x \in X$, where π is Koopman representation, $\mathcal{H}$ is the groupoid representation associated to the action of G on (X, μ), $\rho_x, x \in X$ are the corresponding quasi-regular representations.

b) if, moreover, (G, X, μ) is hyperfinite, then

$$sp(\pi(m)) = sp(\mathcal{H}(m)).$$

c) if, in addition to the conditions of a), μ is G-invariant and non-atomic, then

$sp(\pi_0(m)) = sp(\mathcal{H}(m))$, where π_0 is the restriction of π onto the orthogonal complement of constant functions in $L^2(X, \mu)$.