Lecture 4

Spectra, self-similarity, Schur complement and random Schroedinger operator.

$G \curvearrowright T$

Γ_n - Schreier graph of action on n-th level

$\xi \in \partial T$

Γ_ξ - infinite Schreier graph

corresponding to the point $\xi \in \partial T$ of the boundary ∂T.

$$\xi = \{v_n\}_{n=1}^{\infty},$$

$$(\Gamma_n, v_n) \xrightarrow{n \to \infty} (\Gamma_\xi, \xi)$$

marked graphs

convergence in the natural topology in the space of marked graphs of fixed degree (it is a compact space)

Gri. – Zuk On the asymptotic spectrum 1999

$$m = \sum_{s \in S} a_g (s + s^{-1}) \in \mathbb{C}[G]$$

$M_m^{(n)}$ – "Markov" – Hecke operator in $\ell^2(\Gamma_n)$ corresponding to m (coefficients a_g play a role of weights on generators). if

$0 < a_s < 1$, $\quad 2 \sum_{s \in S} a_s = 1$

get a Markov operator of random walk on Γ_n.

$M_m^{(\xi)}$ – "Markov" Hecke operator in $\ell^2(\Gamma_\xi)$

μ_n – spectral measure of (Γ_n, v_n)

μ — // — (Γ_ξ, ξ)

Proposition $\boxed{\mu_n \xrightarrow{n \to \infty} \mu}$

in $*$-weak topology

$\Rightarrow$ Can study spectra of infinite graphs by spectra of sequences of finite graphs. (covering converging sequences).

$$(\Gamma_1, v_1) \leftarrow (\Gamma_2, v_2) \leftarrow \cdots \leftarrow (\Gamma_n, v_n) \leftarrow (\Gamma_{n+1}, v_{n+1}) \leftarrow$$
$$\leftarrow \cdots \leftarrow (\Gamma_\xi, \xi)$$

in the case Γ_ξ is amenable

$$\boxed{sp(\Gamma_\xi) = \overline{\bigcup_{n=1}^{\infty} sp(\Gamma_n)}}$$

or more generally

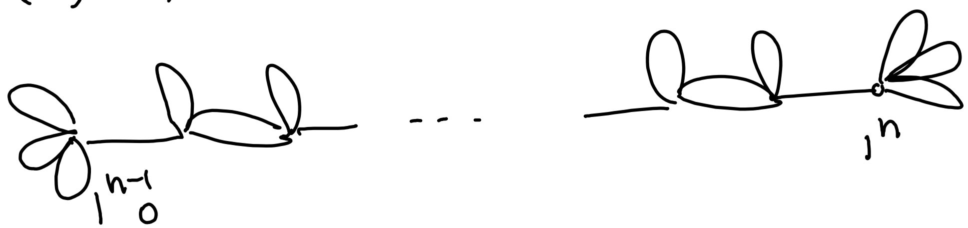
$$\boxed{sp(M_m) = \overline{\bigcup_n sp(M_m^{(n)})}}$$

if $\Gamma, \{\Gamma_n\}$ are associated with the action of self-similar group we can use (sometimes) self-similarity and Schur complement to compute spectra of Γ_n and hence of Γ.

Examples. $\mathcal{G} = \langle a, b, c, d \rangle$

$a = \sigma$, $b = (a, d)$, $c = (a, d)$, $d = (1, b)$ – recursions

Γ_n



Γ_ξ 

one ended

if $\xi \in O(1^\infty)$

Γ_ξ 

two ended

if $\xi \notin O(1^\infty)$

The graphs Γ_ξ are amenable, two ended graphs are 2-periodic. (if considered without labels).

(hyperfinite equivalence relation)

$\sim_y$-orbits are classes of cofinality equivalence relation.

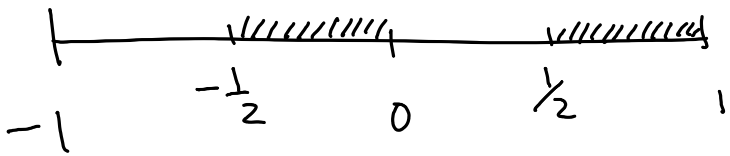
$\xi \sim \eta$ if $\exists N$ s.t. $\xi_n = \eta_n$, $\forall n > N$

$\xi = \{\xi_n\}, \eta = \{\eta_n\}$ (ties are the same)

$Sp(\Gamma_\xi)$ can easily be computed using classical methods (for instance Floquet theory)

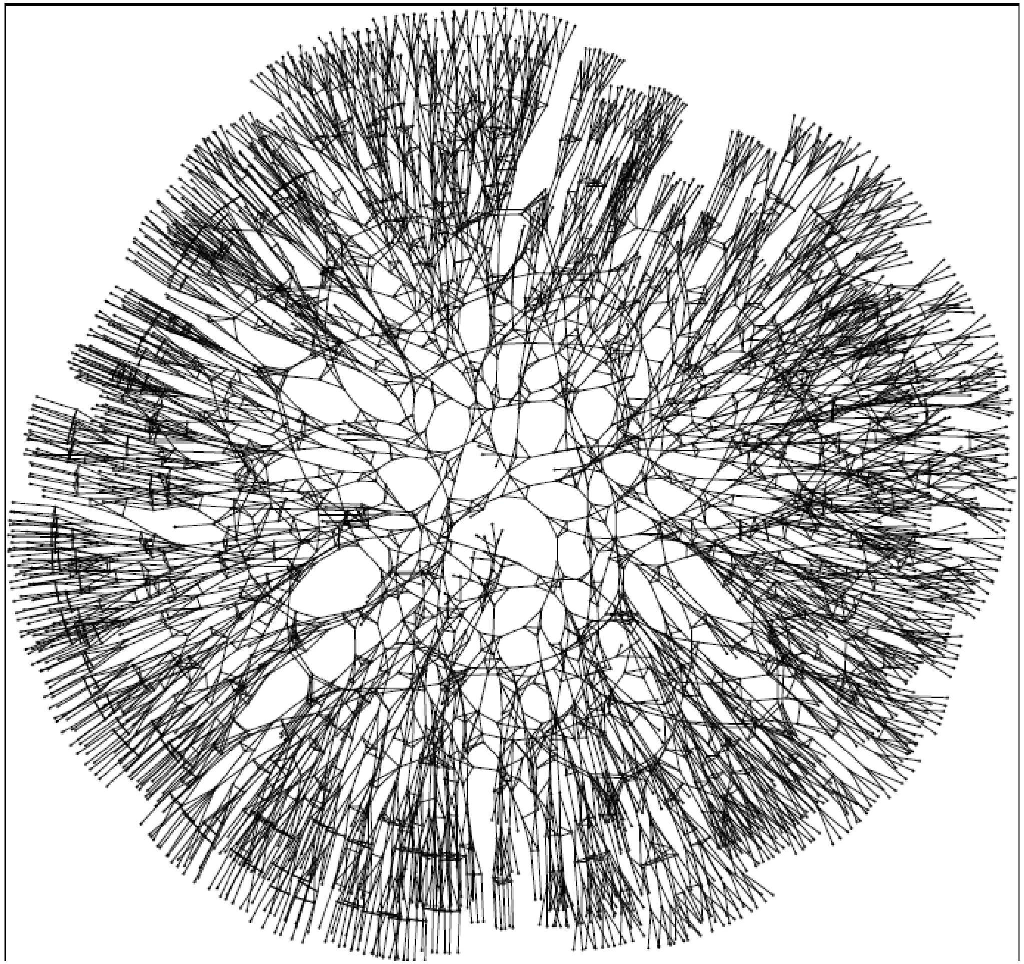
it is $[-2,0] \cup [2,4]$ in the case of equal weights: $w(a) = w(b) = w(c) = w(d) = 1$

[Or $sp(M_{\xi}) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$ if weights are uniform probability distribution].



Surprisingly the spectrum of Cayley graph of $\mathcal{G}$ is the same! (Observed recently by Dudko & Gri).

Cayley graph of $\mathcal{G}$



In general if $w(b) = w(c) = w(d)$ then $Sp(M_m)$ is union of two intervals (or interval, or two points). if

$$m = \lambda a + b + c + d, \qquad 0 < \lambda < 2$$

then

$$sp(M_m) = [-\lambda - 1, \lambda - 1] \cup [3 - \lambda, 3 + \lambda]$$

But if $w(b) = w(c) = w(d)$ does not hold then $Sp(M_m)$ is a Cantor set

D. Lenz, T. Nagnibeda, Gri. !!! (will be explained)

Computation of spectra.

$$m = ta + ub + vc + wd \in \mathbb{C}[\mathcal{G}]$$

$$M_m - E\, I$$

spectral parameter

$$\pi : \mathcal{G} \longrightarrow U(L^2(\partial T, \mu))$$ – Koopman repres.

$$\pi_n : \mathcal{G} \longrightarrow U(\ell^2(V_n))$$ – permutational

representations for $\mathcal{G} \hookrightarrow V_n$ – n-th level of tree

$$\begin{array}{ccc} sp(\mathfrak{I}(m)) & = & \overline{\bigcup_{n=0}^{\infty} \underbrace{sp(\mathfrak{I}_n(m))}_{=\, sp(\Gamma_n)}} \\ \| & & \\ sp(\Gamma_\xi) & & \end{array}$$

Schreier graphs with weights on edges given by m.

Consider

$$R(\lambda,\mu) = \mathfrak{I}(\underbrace{-\lambda a + b + c + d}_{=\, m} - \underbrace{(\mu+1)}_{=\, E} I)$$

$$R_n(\lambda,\mu) = \mathfrak{I}_n(-\lambda a + b + c + d - (\mu+1) I)$$

$$R_n(\lambda,\mu) = -\lambda a_n + b_n + c_n + d_n - (\mu+1) I_n$$

$2^n \times 2^n$ identity matrix.

$$a_n = \pi_n(a), \ \dots, \ d_n = \pi_n(a)$$

$2^n \times 2^n$ matrices,

$$a_0 = b_0 = c_0 = d_0 = 1,$$

$$a_{n+1} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad b_{n+1} = \begin{pmatrix} a_n & 0 \\ 0 & c_n \end{pmatrix}$$

$$c_{n+1} = \begin{pmatrix} a_n & 0 \\ 0 & d_n \end{pmatrix}, \quad d_{n+1} = \begin{pmatrix} 1_n & 0 \\ 0 & b_n \end{pmatrix}$$

matrix recursions

$$R_{n+1}(\lambda,\mu) = \begin{pmatrix} 2a_n - 1 & -\lambda \\ -\lambda & b_n + c_n + d_n - \mu - 1 \end{pmatrix}$$

Schur complement

$$H = H_1 \oplus H_2$$

Hilbert space

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

operator in H.

assume D or A is invertible

$$S_1(M) = A - BD^{-1}C, \qquad S_2(M) = D - CA^{-1}B$$

first and second Schur complements

Th. Suppose that D is invertible. Then M is invertible if and only if $S_1(M)$ is invertible.

Similarly, if A is invertible, then M is invertible if and only if $S_2(M)$ is invertible.

The inverse is computed then by the formula

$$M^{-1} = \begin{pmatrix} S_1^{-1} & -S_1^{-1} B D^{-1} \\ -D^{-1} C S_1^{-1} & D^{-1} C S_1^{-1} B D^{-1} + D^{-1} \end{pmatrix}$$

where $S_1 = S_1(M)$ (similar for $S_2(M)$)

Let

$$\Sigma_1 = \{ (\lambda, \mu) : R(\lambda, \mu) \text{ is not invertible} \}$$

$$\Sigma_{1n} = \{ (\lambda, \mu) : R_n(\lambda, \mu) \text{ is not invertible} \}$$

Define $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ (can be extended to $\mathbb{C}^2$)

$$F : \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \longrightarrow \begin{pmatrix} 2(4-\mu^2) \\ -\mu - \frac{\mu(4-\mu^2)}{\lambda^2} \end{pmatrix}$$

$$G : \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{2\lambda^2}{4-\mu^2} \\ \mu + \frac{\mu\lambda^2}{4-\mu^2} \end{pmatrix}$$

Th. (Barthold- Gri. 2000, Gri - Nekrashevych 2007)

(i) The maps F and G (seen as maps on the projective space) are respectively the first and the second Schur maps restricted on the pencil $R(\lambda,\mu)$.

(ii) The set Σ is invariant with respect to F and G, i.e., $F^{-1}(\Sigma)=\Sigma$, $G^{-1}(\Sigma)=\Sigma$ and relations

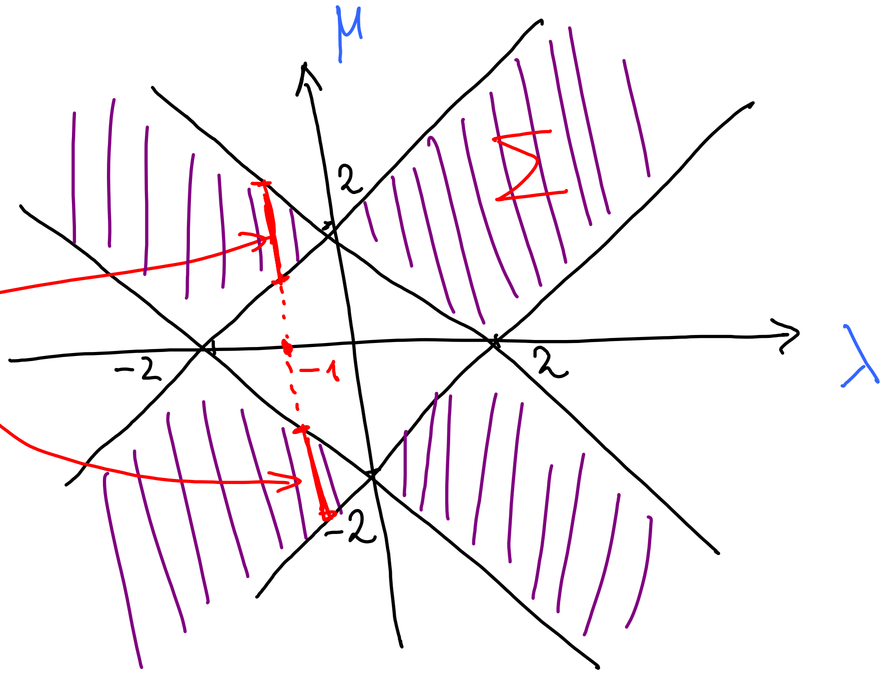
$$\Sigma_{n+1} = F^{-1}(\Sigma_n) = G^{-1}(\Sigma_n)$$

hold.

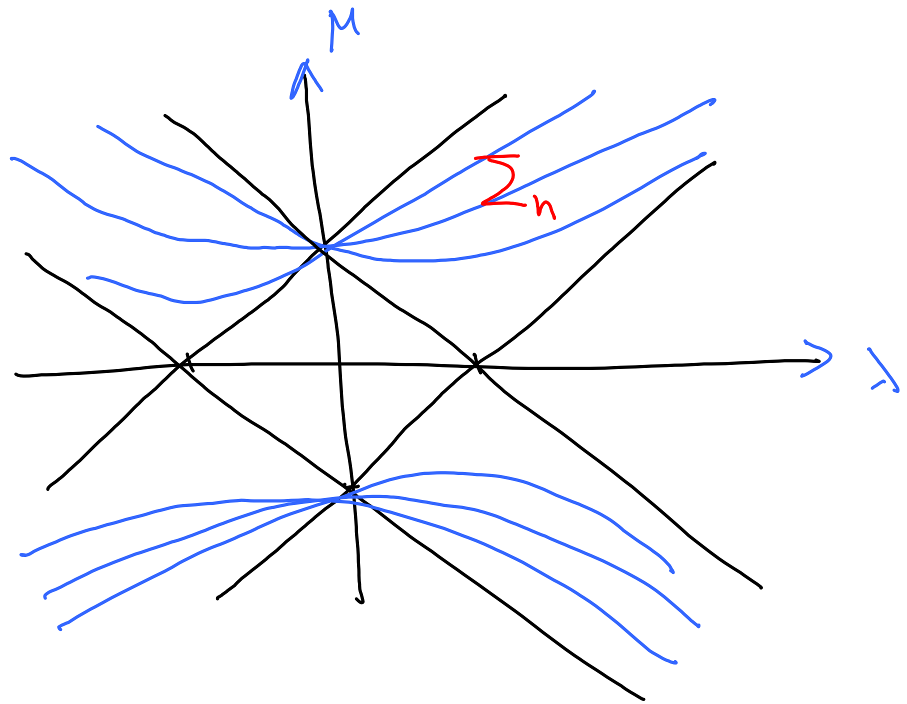
(iii) The set Σ_n is the union of the line

$\lambda + \mu - 2 = 0$ and hyperbolas $H_\theta = 0$, $\theta \in \bigcup_{i=0}^{n-1} \alpha^{-i}(0)$

where $H_\theta = 4 - \mu^2 + \lambda^2 + 4\lambda\theta$.



$m = a+b+c+d$



$$R(\lambda,\mu) = \begin{pmatrix} 2a-\mu & -\lambda \\ -\lambda & b+c+d-\mu-1 \end{pmatrix}$$

omit writing π.

$$t = \frac{b+c+d-1}{2} \text{ is an } \textit{idempotent}, \; t^2 = 1$$

$$(b+c+d-\mu-1)^{-1} = (2t-\mu)^{-1} = \frac{2t+\mu}{4-\mu^2}$$

$$S_1(R(\lambda,\mu)) = 2a - \mu - \frac{\lambda^2(b+c+d+\mu-1)}{4-\mu^2}$$

proportional to

$$-\lambda' a + b + c + d - (\mu'+1)I$$

for $(\lambda', \mu') = F(\lambda, \mu)$

F and G are semi-conjugate to $x \to 2x^2-1$

(Tchebyshev-von Neumann-Ulam map) via

$$\Psi_F(\lambda,\mu) = \frac{4-\mu^2+\lambda^2}{4\lambda}, \quad \Psi_G(\lambda,\mu) = \frac{4-\lambda^2+\mu^2}{4\mu}$$

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "F"] \arrow[d, "\Psi_F"'] & \mathbb{C}^2 \arrow[d, "\Psi_F"] \\
\mathbb{C} \arrow[r, "2x^2-1"] & \mathbb{C}
\end{tikzcd}$$

For 5-parametric pencil

$$R(x,y,z,u,v) = xa + yb + zc + ud + v\,1$$

the Schur maps are:

$$\widetilde{S}_1 : \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} z+y \\ \dfrac{x^2(2yzv-u(y^2+z^2-u^2+v^2))}{(y+z+u+v)(y+z-u-v)(y-z+u-v)(-y+z+u-v)} \\ \dfrac{x^2(2zuv-y(-y^2+z^2+u^2+v^2))}{(y+z+u+v)(y+z-u-v)(y-z+u-v)(-y+z+u-v)} \\ \dfrac{x^2(2yuv-z(y^2-z^2+u^2+v^2))}{(y+z+u+v)(y+z-u-v)(y-z+u-v)(-y+z+u-v)} \\ u+v+\dfrac{x^2(2yzu-v(y^2+z^2+u^2-v^2))}{(y+z+u+v)(y+z-u-v)(y-z+u-v)(-y+z+u-v)} \end{pmatrix}$$

$$\widetilde{S}_2 : \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \dfrac{x^2(y+z)}{(u+v+y+z)(u+v-y-z)} \\ u \\ y \\ z \\ v - \dfrac{x^2(u+v)}{(u+v+y+z)(u+v-y-z)} \end{pmatrix}$$

$\Rightarrow$ joint spectrum

$\Sigma = \{ (x,y,z,u,v) : R(x,y,z,u,v)$ is not invertible$\}$

is invariant set with respect to these two rational maps.

The intersection of Σ by lines in some directions is interval or union of two intervals, in other directions it is a Cantor set

$\Rightarrow$ Σ is "strange" set (strange attractor?)

Change notations: $(x, y, z, u) \longrightarrow (t, u, v, w)$,

$v \longrightarrow -E$ - spectral parameter

Define set

$$P = \{(t, u, v, w) \in \mathbb{R}^4 : t = 0,\ u+v \neq 0,\ u+w \neq 0,\ v+w \neq 0\}$$

M_ξ - "Laplacian" ("Markov") operator on $\Gamma_\xi, \xi \in \partial T$

given by $\quad ta + ub + vc + wd \in \mathbb{R}[\tilde{\mathcal{J}}]$

Theorem (Intervals vs Cantor spectrum for M_ξ)

Let $(t, u, v, w) \in P$ be given and let $\Sigma = \Sigma(t, u, v, w)$

be the spectrum of the associated family of Laplacians $M_\xi(t,u,v,w)$, $\xi \in \partial T \setminus O_w(1^\infty)$. Then, the following holds:

(a) if $u=v=w$ then Σ consists of one or two closed non-trivial intervals and spectral measures are absolutely continuous.

(b) if $u=v=w$ does not hold then Σ is a Cantor set of Lebesgue measure ~~zero~~ and no spectral measure is absolutely continuous.

Part (b) due to D. Lenz, T. Nagnibeda, Gri. 2014-15.

Back to Igor Lysenok's presentation

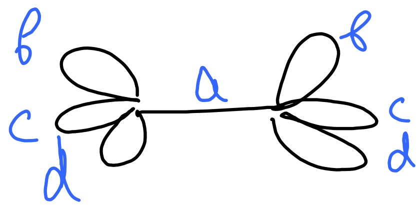
$$\mathcal{G} = \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 = bcd = \sigma^k((ad)^4)$$

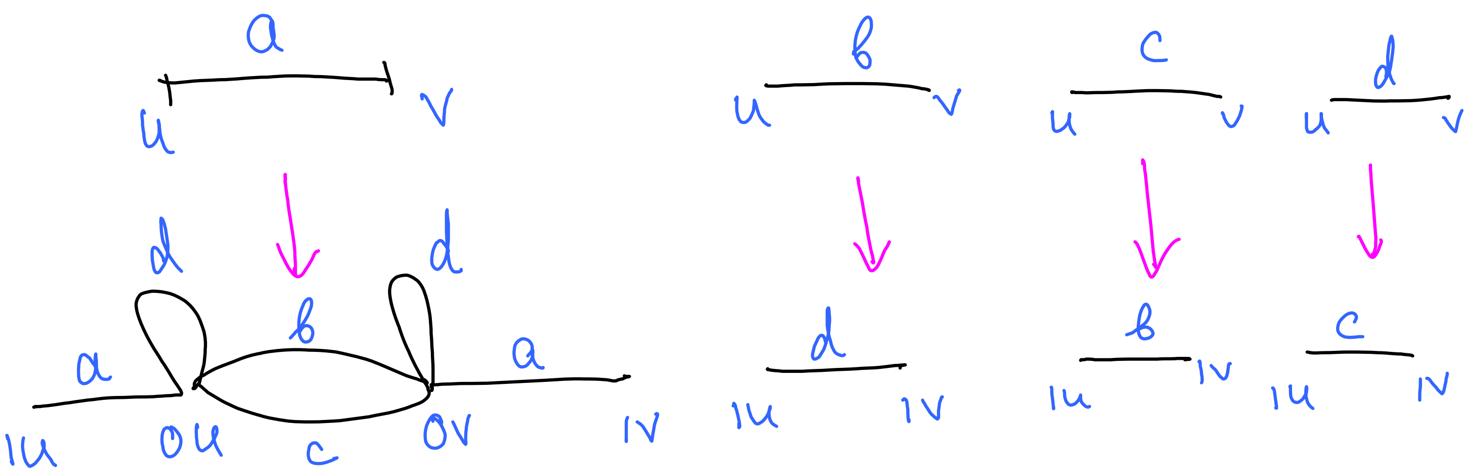
$$= \sigma^k((adacac)^4), \; k = 0, 1, 2, \ldots \rangle$$

$$\sigma : \begin{cases} a \longrightarrow aca \\ b \longrightarrow d \\ c \longrightarrow b \\ d \longrightarrow c \end{cases} \qquad \begin{aligned} aca &= (d, \underline{\underline{a}}) \\ d &= (1, \underline{\underline{b}}) \\ b &= (a, \underline{\underline{c}}) \\ c &= (a, \underline{\underline{d}}) \end{aligned} \qquad \pi_1 \sigma = id$$

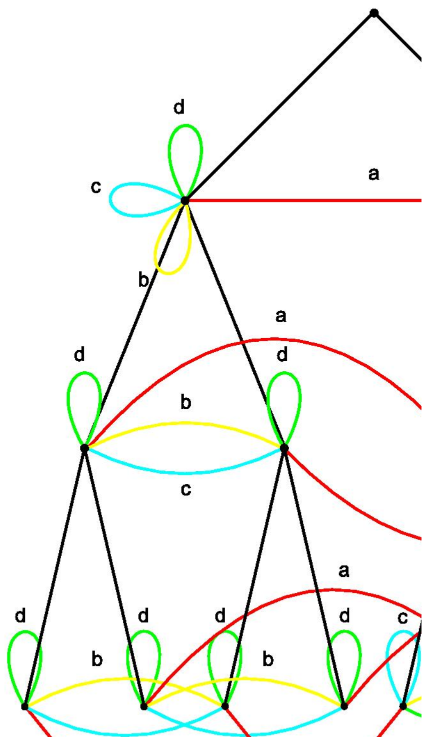
substitution

Sequence $\{\Gamma_n\}$ is a substitutional sequence

$\Gamma_1 =$ 



substitution for graphs (analogue of Lysenok's substitution)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Reduction of the spectral problem for graphs Γ_ξ, $\xi \in \partial T$ to the spectral problem of random Schroedinger.

$$\mathcal{A} = \{a, x, y, z\} \text{ - alphabet}$$

$$\tau : \begin{cases} a \to axa \\ x \to y \\ y \to z \\ z \to x \end{cases}$$

τ is inverse of Lysenoks substitution.

$$\eta = \lim \tau^n(a) \text{ - fixed point}$$

$\Omega_\tau \subset \mathcal{A}^{\mathbb{Z}}$

(Ω_τ, T) - subshift.

τ is not primitive but can be replaced by primive ξ: $a \to ax$, $x \to ay$, $y \to az$, $z \to ax$ that has the same fix point $\Rightarrow$ generate the same subshift.

$\Rightarrow (\Omega_\tau, T)$ is linearly repetitive ($\Rightarrow$ uniquely ergodic and minimal)

+ many other nice properties.

$$D := u + v + w$$

$$f, g : \Omega_{\tilde{z}} \longrightarrow \mathbb{R}$$

$$f(\omega) = \begin{cases} t & \text{if } \omega_0 = a \\ D - w & \text{if } \omega_0 = x \\ D - v & \text{if } \omega_0 = y \\ D - u & \text{if } \omega_0 = z \end{cases}$$

$$g(\omega) = \begin{cases} w & \text{if } \omega_0\omega_1 \in \{ax, xa\} \\ v & \text{if } \omega_0\omega_1 \in \{ay, ya\} \\ u & \text{if } \omega_0\omega_1 \in \{az, za\} \end{cases}$$

$$P = \{ (t, u, z, w) \in \mathbb{R}^4 \mid t \neq 0, u+z \neq 0, u+w \neq 0, z+w \neq 0 \}$$

- if $u = v = w$, then (f, g) is periodic
- if $u = v = w$ does not hold, then (f, g) is not periodic
- Th function f does not vanish anywhere $\Longleftrightarrow (t, u, z, w) \in P$.

Th. (intervals vs Cantor spectrum) Let $(t,u,v,w) \in P$ and f, g be as above and let Σ be the spectrum of the associated family of Schroedinger operators:

$$\{H_\omega\}_{\omega \in \Omega_\tau} \qquad H_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$$

$$\boxed{(H_\omega \varphi)(n) = f(T^n \omega)\varphi(n-1) + f(T^{n+1}\omega)\varphi(n+1) + g(T^n\omega)\varphi(n)}$$

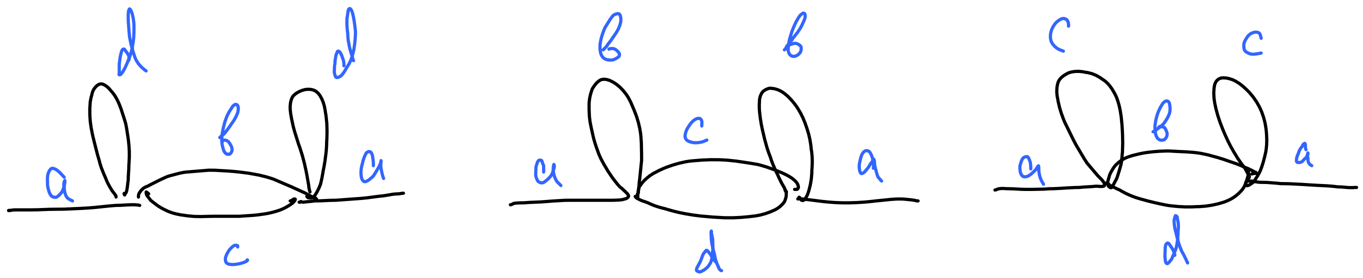
Then

(a) if $u = v = w$ holds then Σ consists of one or two closed non-trivial intervals and all spectral measures are absolutely continuous.

(b) if $u = v = w$ does not hold then Σ is a Cantor set of Lebesgue measure zero and no spectral measure is absolutely continuous.

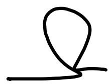
(c) Let $\xi \in \partial T \setminus O_{\gamma}(1^{\infty})$. Then, there exists an $\omega \in \Omega_{\tau}$ such that H_ω is unitarily equivalent to $M_\xi(t, u, v, w)$.

Γ_ξ locally are



the coding

$x \longleftrightarrow \begin{pmatrix} b \\ c \end{pmatrix}$ $\quad y \longleftrightarrow \begin{pmatrix} c \\ d \end{pmatrix}$ $\quad z \longleftrightarrow \begin{pmatrix} b \\ d \end{pmatrix}$

leads to the coding of graphs Γ_ξ by points from  τ and this coding conjugates the operators M_ξ and H_ω.