Geometric models for reducible or hyperbolic substitutions

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Pisot substitutions

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$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$
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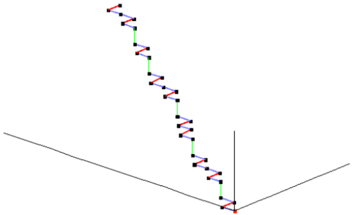

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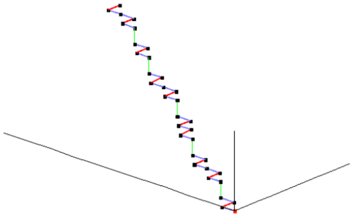

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f(x) = x^3 - x^2 - x - 1$$

$\beta > 1$ Pisot root of $f(x)$: $|\beta'| < 1$, $\forall\, \beta'$ Galois conjugate of β

σ is an irreducible unimodular **Pisot** substitution.

The Rauzy fractal

M_σ-invariant decomposition: $\mathbb{R}^3 = E^u \oplus E^s \cong \mathbb{R} \oplus \mathbb{C}$.

Broken line (balanced): $\sigma^\infty(1) = \epsilon 1213121121312121312112 13\cdots$.

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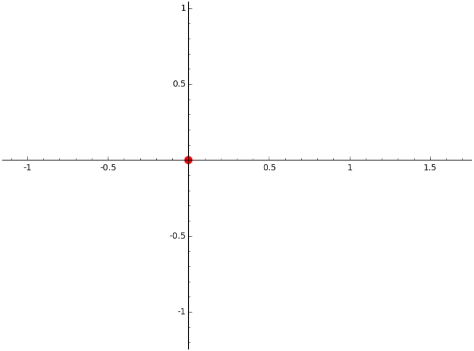
$\mathcal{R} = \bigcup_{i \in \mathcal{A}} \mathcal{R}(i)$ where $\mathcal{R}(i) = \overline{\{\pi_s(\mathbf{l}(p)) : pi \text{ prefix of } \sigma^\infty(1)\}} \subset E^s$.



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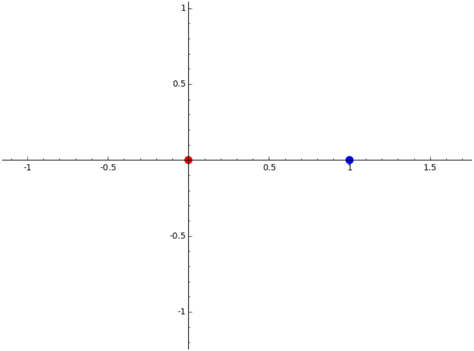

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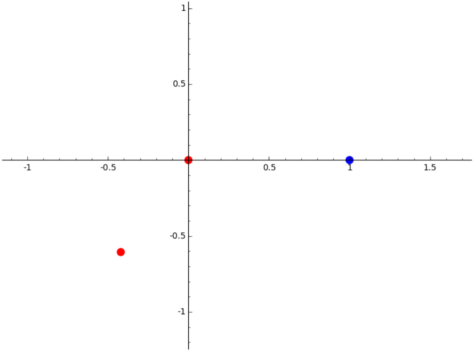

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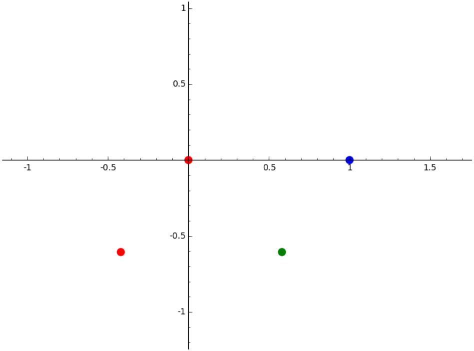

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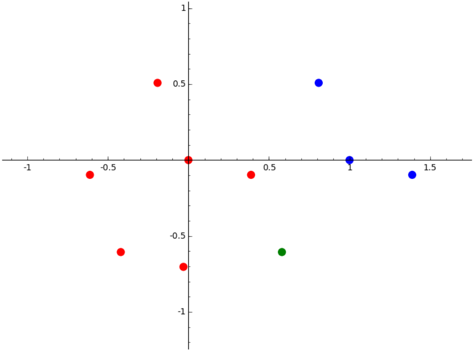

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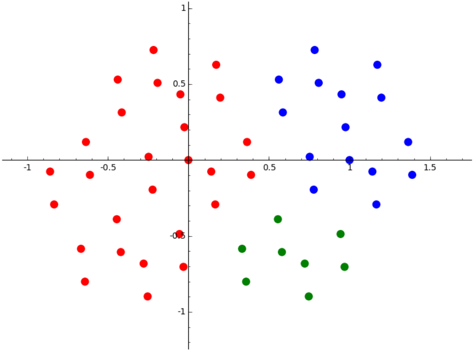

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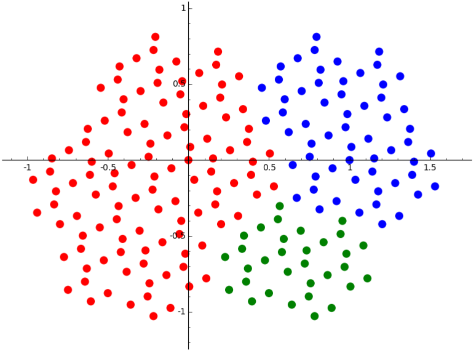

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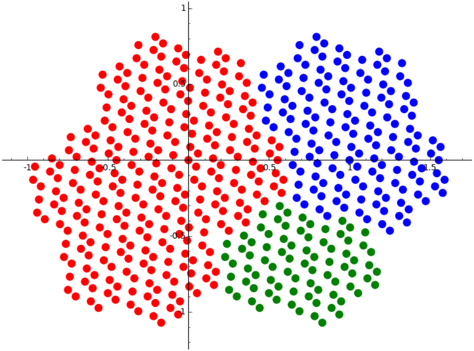

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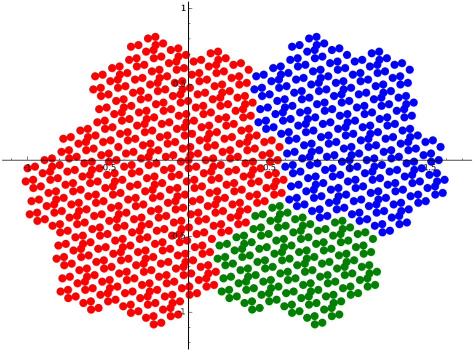

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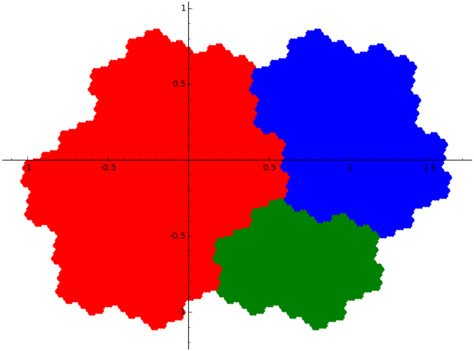

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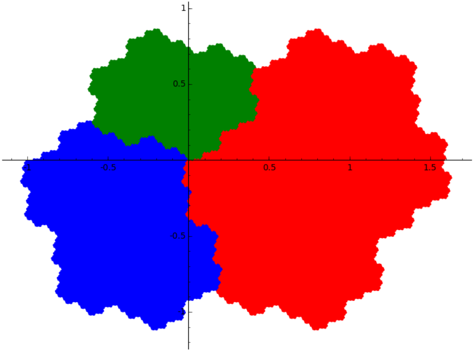

Domain exchange $\mathcal{E} : \mathcal{R}(i) \mapsto \mathcal{R}(i) + \pi_c(\mathbf{e}_i)$.

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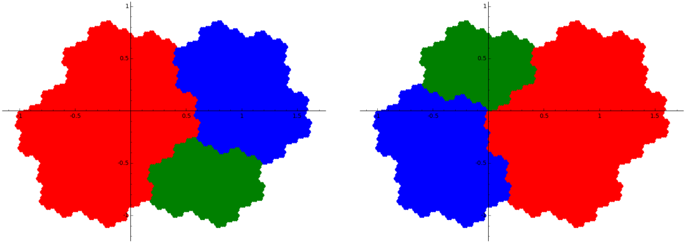

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Strong coincidence condition: $\forall\,(i,j) \in \mathcal{A}^2$, $\exists\, n$, $\exists a \in \mathcal{A}$ such that $\sigma^n(i) = p_1 a s_1$, $\sigma^n(j) = p_2 a s_2$ with $|p_1| = |p_2|$.

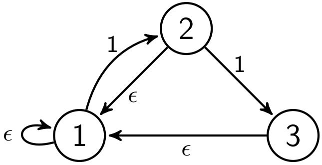
GIFS and dual substitution

Rauzy fractals

- are compact with non-zero measure.
- are the closure of their interior.
- have fractal boundary with zero measure.
- are self-similar, they obey to certain set equations.

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

Prefix graph:



Graph-directed iterated function system (GIFS):

$$\mathcal{R}(a) = \bigcup_{b \xrightarrow{p} a} M_\sigma\, \mathcal{R}(b) + \pi_s(\mathbf{l}(p))$$

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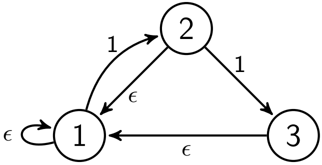


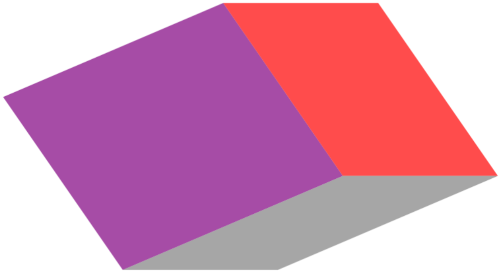
Graph-directed iterated function system (GIFS):

$$\mathcal{R}(1) = M_\sigma\, \mathcal{R}(1) \cup M_\sigma\, \mathcal{R}(2) \cup M_\sigma\, \mathcal{R}(3)$$
$$\mathcal{R}(2) = M_\sigma\, \mathcal{R}(1) + \pi_s(\mathbf{e}_1)$$
$$\mathcal{R}(3) = M_\sigma \mathcal{R}(2) + \pi_s(\mathbf{e}_1)$$

Dual substitution

Dual action on $(d-1)$-dimensional faces:

$$\begin{aligned} \mathbf{E}_1^*(\sigma) : [\mathbf{x}, 1] &\mapsto [M_\sigma^{-1}\mathbf{x}, 1] \cup [M_\sigma^{-1}\mathbf{x}, 2] \cup [M_\sigma^{-1}\mathbf{x}, 3] \\ [\mathbf{x}, 2] &\mapsto [M_\sigma^{-1}(\mathbf{x} + \pi_s(\mathbf{e}_1)), 1] \\ [\mathbf{x}, 3] &\mapsto [M_\sigma^{-1}(\mathbf{x} + \pi_s(\mathbf{e}_1)), 2] \end{aligned}$$



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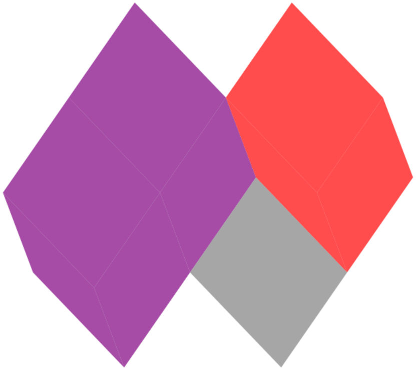

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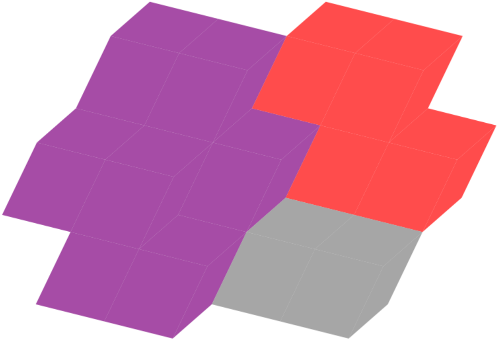

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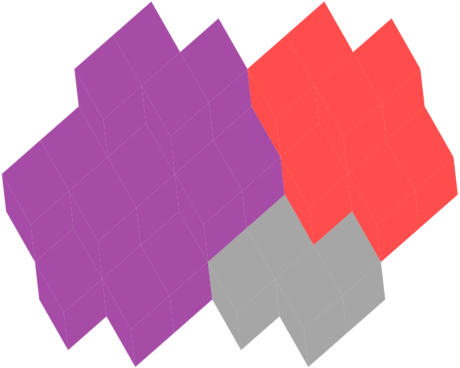

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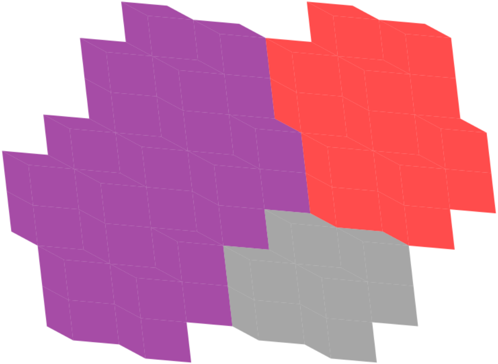

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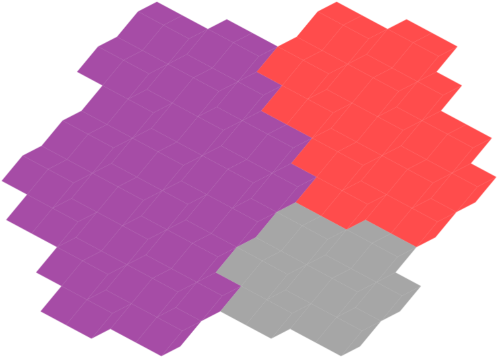

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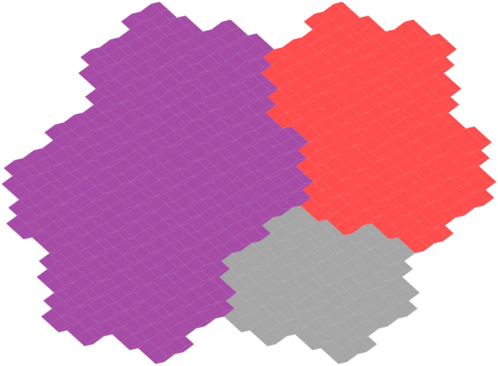

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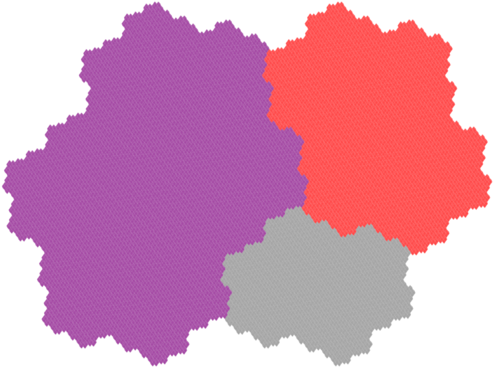

$$\mathcal{R}(i) = \lim_{n\to\infty} \pi_s(M_\sigma^n\, \mathbf{E}_1^*(\sigma)^n([\mathbf{0}, i]))$$

Stepped surfaces

Set of coloured points "near" to E^s:

$$\Gamma = \{(\mathbf{x}, a) \in \mathbb{Z}^d \times \mathcal{A} : \mathbf{x} \in (E^s)^{\geq}, \mathbf{x} - \mathbf{e}_a \in (E^s)^{<}\}$$

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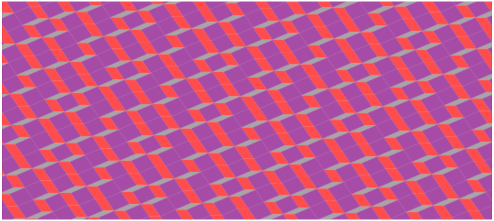
- $\mathbf{E}_1^*(\sigma)(\Gamma) = \Gamma \ \rightarrow$ self-replicating property.
- Aperiodic translation set (Delone set) for a self-replicating multiple tiling made of Rauzy fractals.
- Geometric representation as an arithmetic discrete model of the hyperplane E^s, whose projection is a polygonal tiling.



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$$\begin{array}{ccccc} X_\sigma & \longrightarrow & \mathcal{R} & \xrightarrow{\cong ?} & E^s/\Lambda \\ \downarrow S & & \downarrow \mathcal{E} & & \downarrow \tau \\ X_\sigma & \longrightarrow & \mathcal{R} & \xrightarrow{\cong ?} & E^s/\Lambda \end{array}$$

The conjugation $(X_\sigma, S) \cong (\mathcal{R}, \mathcal{E})$ can be extended to any irreducible unit Pisot substitution satisfying the strong coincidence condition (Arnoux, Ito 2001).

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- For a primitive σ, (X_σ, S) is a minimal, uniquely ergodic, zero entropy subshift.
- Representation map $\varphi : X_\sigma \to \mathcal{R}$, $w_0 w_1 \cdots \mapsto \bigcap_{n \geq 0} \mathcal{R}(w_0 \cdots w_n)$.
- Action of S is conjugate to the domain exchange $\mathcal{E}$.
- $\mathcal{R}$ induces a periodic tiling of E^s ?

Pisot conjecture

Let σ be an irreducible unit Pisot substitution. Then (X_σ, S) has pure discrete spectrum, or equivalently it is metrically isomorphic to a translation on a torus $\mathbb{T}^{d-1}$.

Beyond irreducibility

Reducible

$\#\mathcal{A} > \deg \beta$, $\mathrm{char}(M_\sigma)$ splits over $\mathbb{Q}$ in a Pisot polynomial and in a neutral one.

$$\mathbb{R}^d = E^u \oplus E^s \oplus E^n$$

[joint works with B. Loridant, and with X. Bressaud, T. Jolivet]

Hyperbolic

We have a stable-unstable splitting and $\dim(E^u) > 1$.

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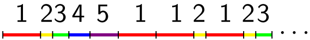
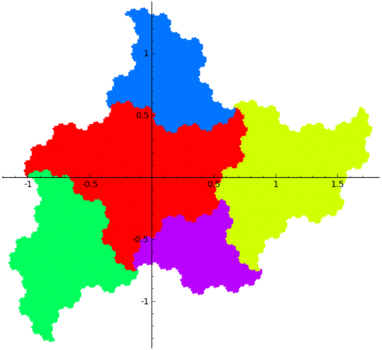
Tool: higher dimensional duals.

Reducibility

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$$

$$\mathrm{char}(M_\sigma) = (x^3 - x - 1)(x^2 - x + 1), \quad \mathbb{R}^5 = E^u \oplus E^s \oplus E^n$$

Projecting the vertices of the broken line 



Problems

Framework: reducible Pisot substitutions.

Some problems:

- Pisot conjecture? False: e.g. Thue-Morse.
- No definition as Hausdorff limit of renormalized patches of polygons.
- No geometric representation for stepped surfaces.
- No periodic (multiple) tiling.

We show some solutions to the last three issues.

Higher dimensional dual maps

Recall: $n = \#\mathcal{A} > d = \deg(\beta)$.

We want to work with $(d-1)$-dimensional faces!
The dual map $\mathbf{E}^*_{n-d+1}(\sigma)$ will suit:

$$\mathbf{E}^*_{n-d+1}(\sigma)(\mathbf{x}, \underline{a})^* = \sum_{\underline{b} \xrightarrow{\underline{p}} \underline{a}} \left(M_\sigma^{-1}(\mathbf{x} - \mathbf{l}(\underline{p})), \underline{b} \right)^*$$

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Remarks:

- $\mathbf{E}^*_{n-d+1}(\sigma)$ acts on $\binom{n}{n-d+1}$ oriented faces.
- If σ is irreducible $n = d$ and $\mathbf{E}^*_{n-d+1}(\sigma) = \mathbf{E}^*_1(\sigma)$.
- $\mathbf{E}_k(\sigma)$ and $\mathbf{E}^*_k(\sigma)$ commute in general with boundary and coboundary operators (Sano, Arnoux, Ito 2001).
- Similar approach for the study of a free group automorphism associated with a complex Pisot root (Arnoux, Furukado, Harriss, Ito 2011).

Stepped surfaces

Let $\mathcal{U} = \{(\mathbf{0}, 2 \wedge 3), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 3 \wedge 4)\}$. We have $\mathcal{U} \subset \mathbf{E}_3^*(\sigma)^5(\mathcal{U})$.
Consider

$$\Gamma_{\mathcal{U}} = \bigcup_{k \geq 0} \mathbf{E}_3^*(\sigma)^{5k}(\mathcal{U})$$

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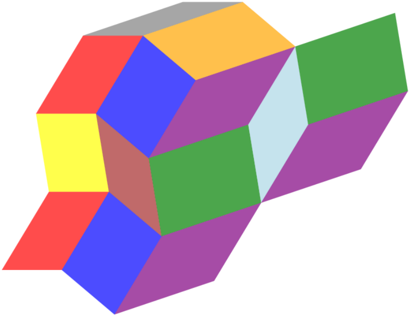

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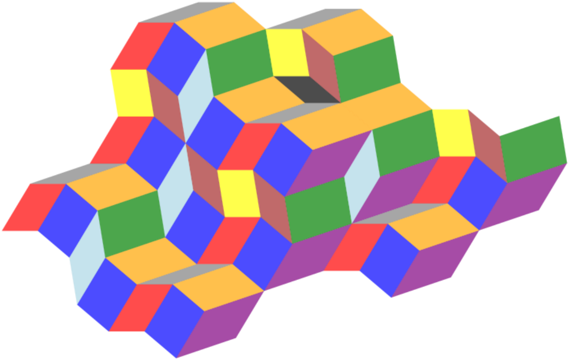

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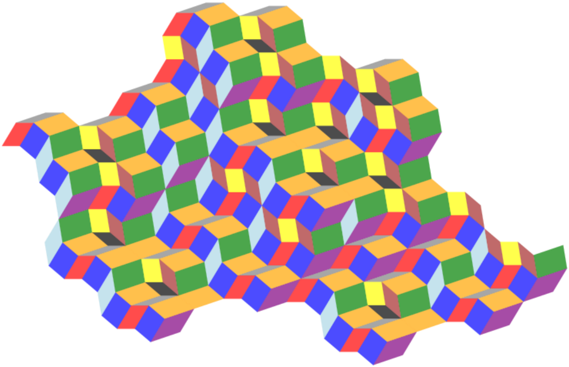

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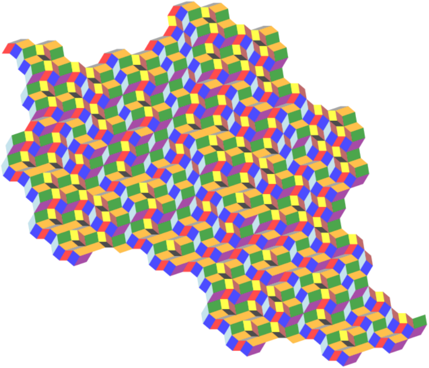

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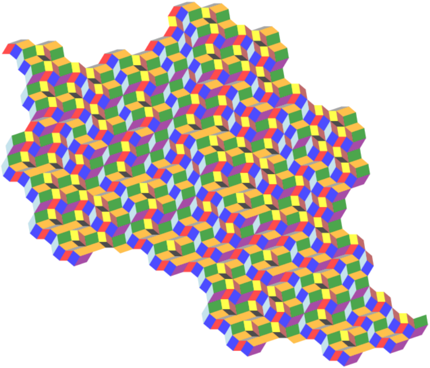

- Projects well: $\mathbf{E}_3^*(\sigma)(\mathbf{0}, \underline{a})^*$ does not overlap, $\forall \underline{a}$.
- Geometric finiteness property: $\pi_s(\Gamma_{\mathcal{U}})$ covers $E^s \cong \mathbb{C}$.
- $\pi_s(\Gamma_{\mathcal{U}})$ is a polygonal tiling.

Rauzy fractals and tilings

Rauzy fractals: $\mathcal{R}(\underline{a}) + \pi_s(\mathbf{x}) = \lim_{k\to\infty} \pi_s(M_\sigma^k\, \mathbf{E}^*_{n-d+1}(\sigma)^k(\mathbf{x}, \underline{a})^*)$.

Properties:

- if neutral polynomial has only roots of modulus one

$$\mathcal{R}(\underline{a}) + \pi_s(\mathbf{x}) = \bigcup_{(\mathbf{y},\underline{b})\in \mathbf{E}^*_{n-d+1}(\sigma)(\mathbf{x},\underline{a})} M_\sigma\big(\mathcal{R}(\underline{b}) + \pi_s(\mathbf{y})\big),$$

  where the union is measure disjoint.

- compact with nonzero measure.
- closure of the interior.
- boundary has zero measure.

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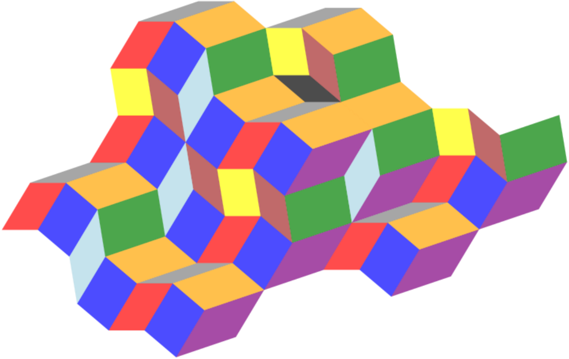
The collection $\{\mathcal{R}(\underline{a}) + \pi_s(\mathbf{x}) : (\mathbf{x}, \underline{a})^* \in \Gamma_{\mathcal{U}}\}$ is a self-replicating tiling.



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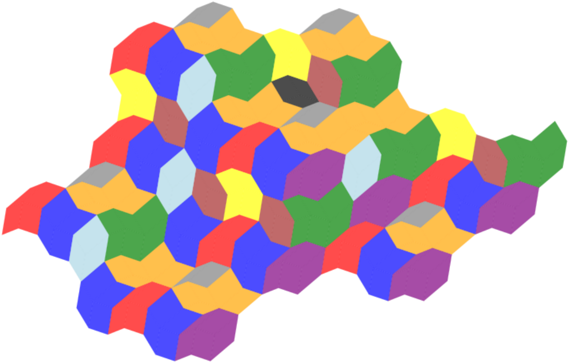

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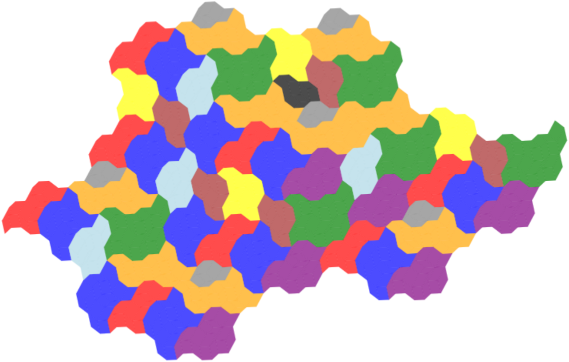

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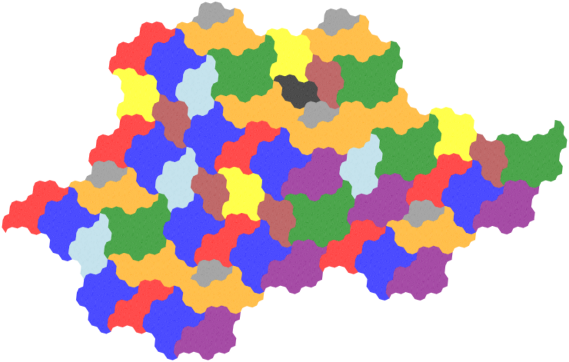

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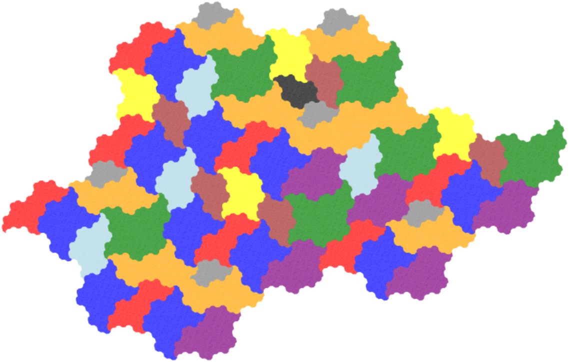

Periodic tilings

Recall: the original Hokkaido tile can not tile periodically (Ei, Ito 2005)

$\mathcal{U} = \{(\mathbf{0}, 2 \wedge 3), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 3 \wedge 4)\}$.

- The patch $\pi_c(\mathcal{U})$ tiles periodically by the lattice $\Lambda_{\mathcal{U}} = \pi_c((\mathbf{e}_4 - \mathbf{e}_3)\mathbb{Z} + (\mathbf{e}_4 - \mathbf{e}_2)\mathbb{Z})$.

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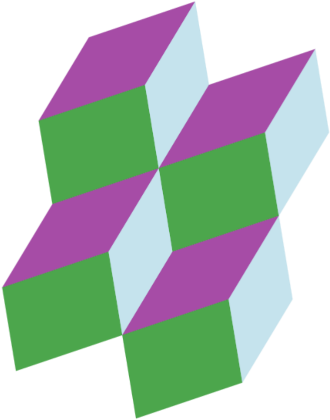

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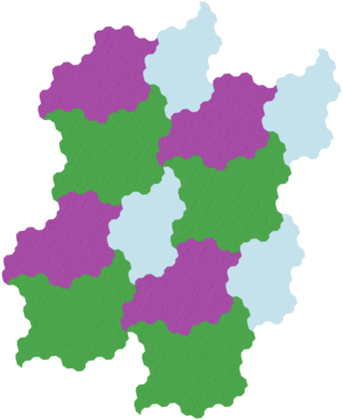

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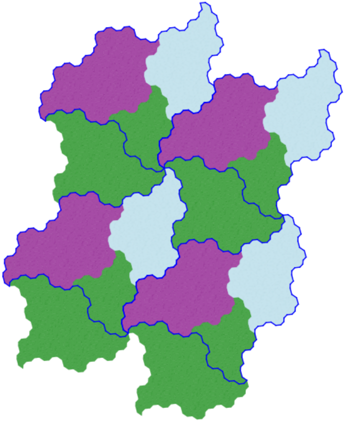

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- $\mathcal{R}_{\mathcal{U}} + \Lambda_{\mathcal{U}}$ is a *periodic tiling*.
- Do you see the original Hokkaido tile?

Broken lines and morphisms

Being reducible means that some linear dependencies arise when we project the basis vectors $\{\mathbf{e}_a\}_{a\in\mathcal{A}}$ from $\mathbb{R}^5$ to $\mathbb{R}^3$ along E^n:

$$\pi(\mathbf{e}_1) = \pi(\mathbf{e}_3) + \pi(\mathbf{e}_4), \quad \pi(\mathbf{e}_5) = \pi(\mathbf{e}_2) + \pi(\mathbf{e}_3)$$

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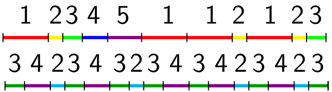
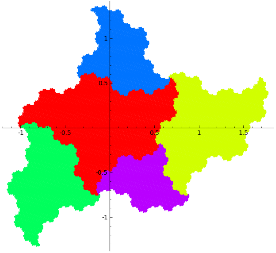
Combinatorially this is equivalent to applying the morphism

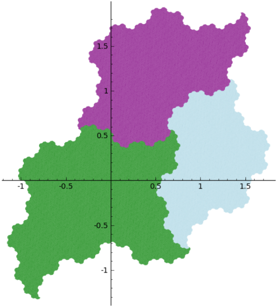
$$\chi : 1 \mapsto 34, \quad 2 \mapsto 2, \quad 3 \mapsto 3, \quad 4 \mapsto 4, \quad 5 \mapsto 32.$$



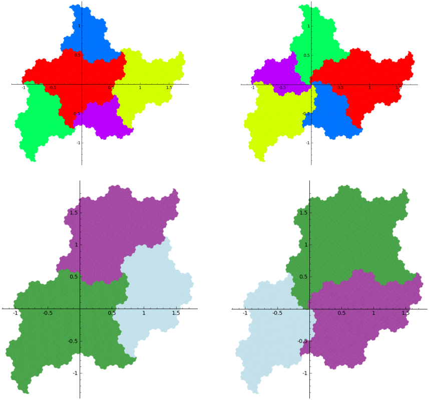
Project now the vertices of the new broken line. . .

Broken lines and morphisms





Domain exchange



- $(\mathcal{T}, \mathcal{E}_{\mathcal{T}})$ is a *domain exchange* on the original Hokkaido tile.

$$\mathcal{E}_{\mathcal{T}} : \mathcal{T}(a) \mapsto \mathcal{T}(a) + \pi_s(\mathbf{e}_a), \ a \in \mathcal{A}$$

- $(\mathcal{R}, \mathcal{E})$ is a *toral translation*, since it induces a periodic tiling of $\mathbb{C}$.

$$\mathcal{E} : \mathcal{R}(a) \mapsto \mathcal{R}(a) + \pi_s(\mathbf{e}_a), \ a \in \{2, 3, 4\}$$

- $\mathcal{E}_{\mathcal{T}}$ is the *first return* of $\mathcal{E}$ on $\mathcal{T}$.

Codings of the domain exchange

Let $\Omega = \overline{\{S^k w : k \in \mathbb{N}\}}$, where $w = \chi(u)$ is the coded fixed point of σ.

We have the following commutative diagram:

$$\begin{array}{ccccccc}
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \mathcal{R} & \longrightarrow & \mathbb{C}/\Lambda \\
\downarrow S & & \downarrow S & & \downarrow E & & \downarrow E \\
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \mathcal{R} & \longrightarrow & \mathbb{C}/\Lambda
\end{array}$$

ϕ measure conjugation.

We can generalize what shown for the family of substitutions

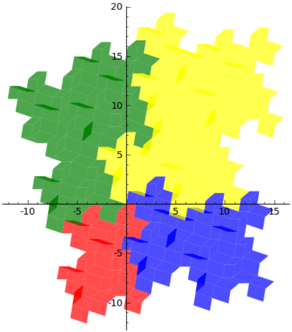
$$\sigma_t : \ 1 \mapsto 1^{t+1}2, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 1^t5, \ 5 \mapsto 1$$

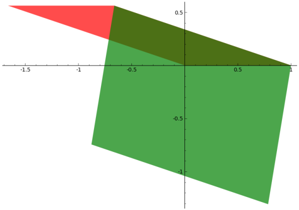
$\to (X_\sigma, S, \mu)$ is the first return of a toral translation.

Remarks

Important hypotheses:

- Projecting well → projection of patches onto E^s behaves well.





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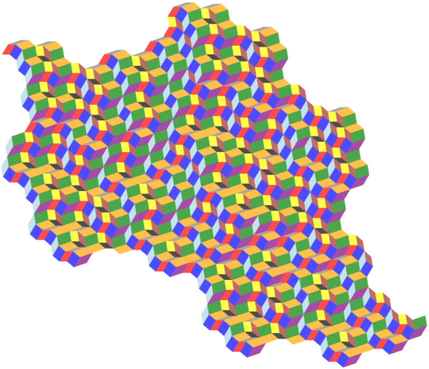
- Geometric finiteness property $\to$ covering property for the stepped surface.
- Roots of the neutral polynomial of modulus one $\to$ measure disjointness in the set equation.
- Positivity: $\bigwedge_{i=1}^{k} M_\sigma$ can have negative entries.
  Can we control cancellation? Can we control it using orientation of faces? For Tribo:

  $$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

  Possible definition of positivity: $|M_k|^j = |M_k^j|$, for all $j \in \mathbb{N}$.

Irreducibilifying

Guiding philosophy: try to turn the substitution into an irreducible one!



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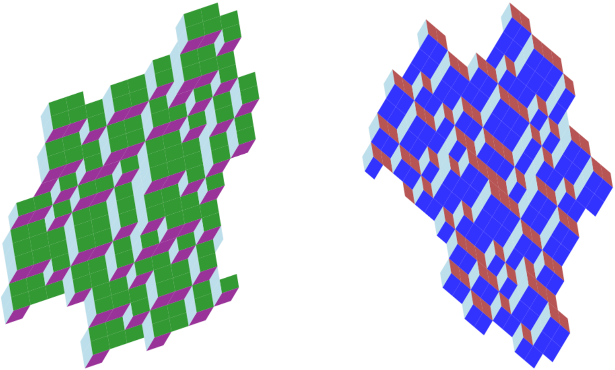

Figure: Changing suitably the projection we get different polygonal tilings by some faces of three different types.

Perspectives

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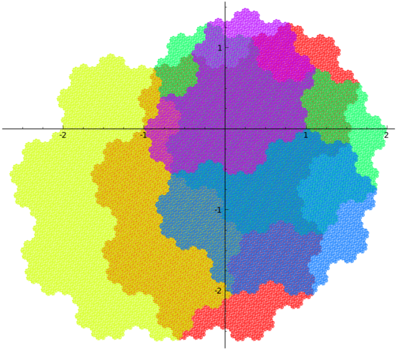
- Height group.
- Pisot conjecture for reducible substitutions?

Strange examples

(Joint with X. Bressaud, T. Jolivet)

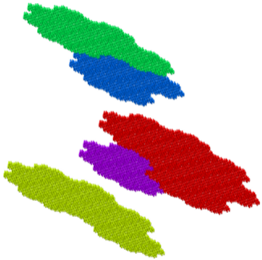
The geometric interpretation seems to get harder for other substitutions, not satisfying the strong coincidence condition:

$$\sigma : 1 \mapsto 213, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 21$$
$$\mathrm{char}(M_\sigma) = (x^2 + x + 1)(x^3 - 2x^2 + x - 1)$$



Lifting in the neutral space

Projection $\pi_{s,n} : \mathbb{R}^d \to E^s \oplus E^n$.



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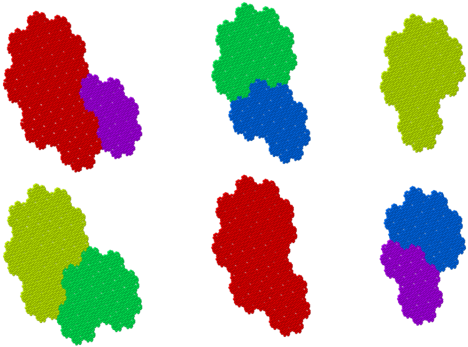

Criterion to know whether we get finitely many layers and NEW strong coincidence condition.

Gluing together

Projecting down suitably we can glue the subtiles together...

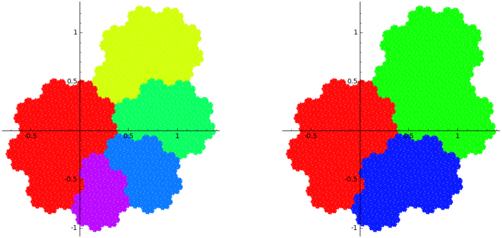

Figure: Symbolic splitting associated with the irreducible substitution $\tau : 1 \mapsto 12, 2 \mapsto 32, 3 \mapsto 1$.

... and obtain the connection with an irreducible substitution.

Philosophy: dynamically the reducible substitutive system behaves exactly as the irreducible one, after identifying some letters / changing projection. Technique: symbolic splitting.

Hokkaido again

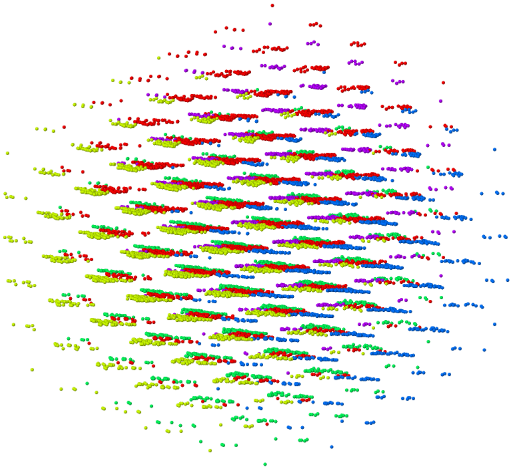

Figure: Rauzy fractal of the Hokkaido substitution in $E^s \oplus E^n$. The points distribute with logarithmic growth on a two-dimensional lattice.

Hyperbolic case

A self-induced IET substitution

$$\sigma : 1 \mapsto 124, 2 \mapsto 1224, 3 \mapsto 124334, 4 \mapsto 12434$$
$$\mathrm{char}(M_\sigma) = x^4 - 7x^3 + 13x^2 - 7x + 1, \quad \beta_1, \beta_2 > 1, \; \beta_3, \beta_4 < 1$$

Geometric representation $\rightarrow$ two fractal windows generated by

$$\mathbf{E}_2(\sigma)(\mathbf{x}, \underline{a}) = \sum_{\underline{a} \xrightarrow{\underline{p}} \underline{b}} \big(M_\sigma \mathbf{x} + \mathbf{l}(\underline{p})), \underline{b}\big)$$
$$\mathbf{E}_2^*(\sigma)(\mathbf{x}, \underline{a})^* = \sum_{\underline{b} \xrightarrow{\underline{p}} \underline{a}} \big(M_\sigma^{-1}(\mathbf{x} - \mathbf{l}(\underline{p})), \underline{b}\big)^*$$

Argue with similar hypotheses as for the reducible case: positivity, projecting-well, geometric finiteness property, etc.

Very complicated to state results in full generality! See Sage...