Geometric models for reducible or hyperbolic substitutions

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Pisot substitutions

\( \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \)

\[ \sigma(1) = 12 \]
Pisot substitutions

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]
\[ \sigma^2(1) = 1213 \]
Pisot substitutions

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]

\[ \sigma^3(1) = 1213121 \]
Pisot substitutions

\[ \sigma: 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]

\[ \sigma^4(1) = 1213121121312 \]
Pisot substitutions

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]

\[ \sigma^5(1) = 12131211213121213121121213 \]
Pisot substitutions

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1 \]
\[ \sigma^\infty(1) = 121312112131212131211213 \cdots \in \{1, 2, 3\}^\mathbb{N} \]
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\[ \sigma^\infty(1) = 12131211213121213121121213 \cdots \in \{1, 2, 3\}^\mathbb{N} \]

\[ M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f(x) = x^3 - x^2 - x - 1 \]

\( \beta > 1 \) Pisot root of \( f(x) : |\beta'| < 1, \ \forall \beta' \text{ Galois conjugate of } \beta \)

\( \sigma \) is an irreducible unimodular Pisot substitution.
The Rauzy fractal

$M_\sigma$-invariant decomposition: $\mathbb{R}^3 = E^u \oplus E^s \cong \mathbb{R} \oplus \mathbb{C}$.

Broken line (balanced): $\sigma^\infty(1) = \epsilon 121312112131212131211211213 \cdots$.

Rauzy fractal

$\mathcal{R} = \bigcup_{i \in \mathcal{A}} \mathcal{R}(i)$ where $\mathcal{R}(i) = \overline{\{ \pi_s(\mathbf{l}(p)) : pi \text{ prefix of } \sigma^\infty(1) \}} \subset E^s$. 
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Domain exchange $E^i: \mathcal{R}(i) \mapsto \mathcal{R}(i) + \pi_c(e_i)$. 

Strong coincidence condition: $\forall (i, j) \in A^2, \exists n, \exists a \in A$ such that $\sigma^n(i) = p_1 a_1, \sigma^n(j) = p_2 a_2$ with $|p_1| = |p_2|$. 

\begin{tikzpicture}[scale=0.8]
  \draw[-stealth] (-1.5,0) -- (1.5,0) node[right] {\hspace{1cm}};
  \draw[-stealth] (0,-1.5) -- (0,1.5) node[above] {\hspace{1cm}};
  \fill[red] (0,0) circle (2pt);
  \fill[blue] (1,0) circle (2pt);
\end{tikzpicture}
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[Diagram of the Rauzy fractal with points marked on a plane]
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![Diagram of the Rauzy fractal]
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Rauzy fractals

- are compact with non-zero measure.
- are the closure of their interior.
- have fractal boundary with zero measure.
- are self-similar, they obey to certain set equations.

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Prefix graph:

Graph-directed iterated function system (GIFS):

\[ \mathcal{R}(a) = \bigcup_{b \xrightarrow{p} a} M_\sigma \mathcal{R}(b) + \pi_s(I(p)) \]
GIFS and dual substitution

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![Prefix graph image]

Graph-directed iterated function system (GIFS):

\[
\begin{align*}
\mathcal{R}(1) &= M_{\sigma} \mathcal{R}(1) \cup M_{\sigma} \mathcal{R}(2) \cup M_{\sigma} \mathcal{R}(3) \\
\mathcal{R}(2) &= M_{\sigma} \mathcal{R}(1) + \pi_s(e_1) \\
\mathcal{R}(3) &= M_{\sigma} \mathcal{R}(2) + \pi_s(e_1)
\end{align*}
\]
Dual substitution

Dual action on \((d - 1)\)-dimensional faces:

\[
\mathbf{E}_1^*(\sigma) : [\mathbf{x}, 1] \mapsto [M_{\sigma}^{-1}\mathbf{x}, 1] \cup [M_{\sigma}^{-1}\mathbf{x}, 2] \cup [M_{\sigma}^{-1}\mathbf{x}, 3]
\]
\[
[x, 2] \mapsto [M_{\sigma}^{-1}(x + \pi_s(e_1)), 1]
\]
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[x, 3] \mapsto [M_{\sigma}^{-1}(x + \pi_s(e_1)), 2]
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\[ \mathcal{R}(i) = \lim_{n \to \infty} \pi_s(M_{\sigma}^n \mathbf{E}_1^*(\sigma)^n([0, i])) \]
Stepped surfaces

Set of coloured points “near” to $E^s$:

$$\Gamma = \{ (x, a) \in \mathbb{Z}^d \times \mathcal{A} : x \in (E^s)^\geq, x - e_a \in (E^s)^< \}$$
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• $E_1^*(\sigma)(\Gamma) = \Gamma \rightarrow$ self-replicating property.

• Aperiodic translation set (Delone set) for a self-replicating multiple tiling made of Rauzy fractals.

• Geometric representation as an arithmetic discrete model of the hyperplane $E^s$, whose projection is a polygonal tiling.
Pisot conjecture

Substitutive system

$(X_\sigma, S)$, where $X_\sigma = \{S^k u : k \in \mathbb{Z}\}$
Pisot conjecture

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- For a primitive \(\sigma,\) \((X_\sigma, S)\) is a minimal, uniquely ergodic, zero entropy subshift.
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- For a primitive \(\sigma\), \((X_\sigma, S)\) is a minimal, uniquely ergodic, zero entropy subshift.
- Representation map \(\varphi : X_\sigma \to \mathcal{R}, w_0 w_1 \cdots \mapsto \bigcap_{n \geq 0} \mathcal{R}(w_0 \cdots w_n)\).
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- \(\mathcal{R}\) induces a periodic tiling of \(E^s\) ?
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### Substitutive system

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\[
\begin{array}{ccc}
X_\sigma & \overset{\sim}{\longrightarrow} & \mathcal{R} \overset{\sim}{\longrightarrow} E^s / \Lambda \\
\downarrow S & & \downarrow \mathcal{E} & \downarrow \tau \\
X_\sigma & \overset{\sim}{\longrightarrow} & \mathcal{R} \overset{\sim}{\longrightarrow} E^s / \Lambda
\end{array}
\]

The conjugation \((X_\sigma, S) \cong (\mathcal{R}, \mathcal{E})\) can be extended to any irreducible unit Pisot substitution satisfying the strong coincidence condition (Arnoux, Ito 2001).
Pisot conjecture

Substitutive system

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- Action of \(S\) is conjugate to the domain exchange \(\mathcal{E}\).
- \(\mathcal{R}\) induces a periodic tiling of \(E^s\) ?

Pisot conjecture

Let \(\sigma\) be an irreducible unit Pisot substitution. Then \((X_\sigma, S)\) has pure discrete spectrum, or equivalently it is metrically isomorphic to a translation on a torus \(\mathbb{T}^{d-1}\).
Beyond irreducibility

Reducible

\#\mathcal{A} > \deg \beta, \ char(M_{\sigma}) \ splits \ over \ \mathbb{Q} \ in \ a \ Pisot \ polynomial \ and \ in \ a \ neutral \ one.

\[ \mathbb{R}^d = E^u \oplus E^s \oplus E^n \]

[joint works with B. Loridant, and with X. Bressaud, T. Jolivet]

Hyperbolic

We have a stable-unstable splitting and \( \dim(E^u) > 1 \).
Beyond irreducibility

Reducible

\#A > \deg \beta, \ \text{char}(M_\sigma) \text{ splits over } \mathbb{Q} \text{ in a Pisot polynomial and in a neutral one.}

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We have a stable-unstable splitting and \( \dim(E^u) > 1 \).

**Tool:** higher dimensional duals.
Reducibility

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 5, \ 5 \mapsto 1 \]

\[ \text{char}(M_\sigma) = (x^3 - x - 1)(x^2 - x + 1), \quad \mathbb{R}^5 = E^u \oplus E^s \oplus E^n \]

Projecting the vertices of the broken line 1 2 3 4 5 1 1 2 1 2 3 \ldots
Some problems:

- Pisot conjecture? False: e.g. Thue-Morse.
- No definition as Hausdorff limit of renormalized patches of polygons.
- No geometric representation for stepped surfaces.
- No periodic (multiple) tiling.

We show some solutions to the last three issues.
Higher dimensional dual maps

Recall: \( n = \# \mathcal{A} > d = \deg(\beta) \).

We want to work with \((d - 1)\)-dimensional faces!

The dual map \( E_{n-d+1}^*(\sigma) \) will suit:

\[
E_{n-d+1}^*(\sigma)(x, a)^* = \sum_{b \overrightarrow{p} a} (M_{\sigma}^{-1}(x - l(p)), b)^*
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\mathbf{E}^*_n(\sigma)(x, a)^* = \sum_{b \stackrel{p}{\longrightarrow} a} (M^{-1}_\sigma(x - l(p)), b)^*
\]

Remarks:

- \( \mathbf{E}^*_n(\sigma) \) acts on \( \binom{n}{n-d+1} \) oriented faces.
- If \( \sigma \) is irreducible \( n = d \) and \( \mathbf{E}^*_n(\sigma) = \mathbf{E}^*_1(\sigma) \).
- \( \mathbf{E}_k(\sigma) \) and \( \mathbf{E}^*_k(\sigma) \) commute in general with boundary and coboundary operators (Sano, Arnoux, Ito 2001).
- Similar approach for the study of a free group automorphism associated with a complex Pisot root (Arnoux, Furukado, Harriss, Ito 2011).
Let $\mathcal{U} = \{(0, 2 \land 3), (0, 2 \land 4), (0, 3 \land 4)\}$. We have $\mathcal{U} \subset E_3^*(\sigma)^5(\mathcal{U})$. 
Consider

$$\Gamma_\mathcal{U} = \bigcup_{k \geq 0} E_3^*(\sigma)^{5k}(\mathcal{U})$$
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Stepped surfaces

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Let \( U = \{(0, 2 \wedge 3), (0, 2 \wedge 4), (0, 3 \wedge 4)\} \). We have \( U \subset E^*_3(\sigma)^5(U) \).

Consider

\[
\Gamma_U = \bigcup_{k \geq 0} E^*_3(\sigma)^{5k}(U)
\]

- Projects well: \( E^*_3(\sigma)(0, a)^* \)
  does not overlap, \( \forall a \).
- Geometric finiteness property:
  \( \pi_s(\Gamma_U) \) covers \( E^s \cong \mathbb{C} \).
- \( \pi_s(\Gamma_U) \) is a polygonal tiling.
Rauzy fractals: $\mathcal{R}(a) + \pi_s(x) = \lim_{k \to \infty} \pi_s(M^k_{\sigma} E_{n-d+1}^*(\sigma)^k(x, a)^*)$.

Properties:

- if neutral polynomial has only roots of modulus one

$$\mathcal{R}(a) + \pi_s(x) = \bigcup_{(y, b) \in E_{n-d+1}^*(\sigma)(x, a)} M_{\sigma}(\mathcal{R}(b) + \pi_s(y)),$$

where the union is measure disjoint.

- compact with nonzero measure.

- closure of the interior.

- boundary has zero measure.
Rauzy fractals: $\mathcal{R}(a) + \pi_s(x) = \lim_{k \to \infty} \pi_s(M_{\sigma}^k E_{n-d+1}(\sigma)^k(x, a)^*)$.

The collection $\{\mathcal{R}(a) + \pi_s(x) : (x, a)^* \in \Gamma_U\}$ is a self-replicating tiling.
Rauzy fractals and tilings

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The collection $\{\mathcal{R}(a) + \pi_s(x) : (x, a)^* \in \Gamma_U\}$ is a **self-replicating tiling**.
Rauzy fractals: \( \mathcal{R}(a) + \pi_s(x) = \lim_{k \to \infty} \pi_s(M^k_{\sigma} E^*_n \sigma^k(x, a)^*). \)

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Recall: the original Hokkaido tile cannot tile periodically (Ei, Ito 2005)

\[ \mathcal{U} = \{(0, 2 \land 3), (0, 2 \land 4), (0, 3 \land 4)\}. \]

- The patch \( \pi_c(\mathcal{U}) \) tiles periodically by the lattice

\[ \Lambda_\mathcal{U} = \pi_c((\mathbf{e}_4 - \mathbf{e}_3)\mathbb{Z} + (\mathbf{e}_4 - \mathbf{e}_2)\mathbb{Z}). \]
Periodic tilings

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- \( \mathcal{R}_\mathcal{U} + \Lambda_\mathcal{U} \) is a periodic tiling.

- Do you see the original Hokkaido tile?
Being reducible means that some linear dependencies arise when we project the basis vectors \( \{ e_a \}_{a \in A} \) from \( \mathbb{R}^5 \) to \( \mathbb{R}^3 \) along \( E^n \):

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\pi(e_1) = \pi(e_3) + \pi(e_4), \quad \pi(e_5) = \pi(e_2) + \pi(e_3)
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Combinatorially this is equivalent to applying the morphism

\[
\chi: 1 \mapsto 34, \quad 2 \mapsto 2, \quad 3 \mapsto 3, \quad 4 \mapsto 4, \quad 5 \mapsto 32.
\]

Project now the vertices of the new broken line...
Broken lines and morphisms
(\mathcal{T}, \mathcal{E}_\mathcal{T}) is a *domain exchange* on the original Hokkaido tile.

\[ \mathcal{E}_\mathcal{T} : \mathcal{T}(a) \mapsto \mathcal{T}(a) + \pi_s(e_a), \ a \in \mathcal{A} \]

(\mathcal{R}, \mathcal{E}) is a *toral translation*, since it induces a periodic tiling of \( \mathbb{C} \).

\[ \mathcal{E} : \mathcal{R}(a) \mapsto \mathcal{R}(a) + \pi_s(e_a), \ a \in \{2, 3, 4\} \]

\( \mathcal{E}_\mathcal{T} \) is the *first return* of \( \mathcal{E} \) on \( \mathcal{T} \).
Let $\Omega = \{S^k w : k \in \mathbb{N}\}$, where $w = \chi(u)$ is the coded fixed point of $\sigma$.

We have the following commutative diagram:

\[
\begin{array}{cccc}
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \mathcal{R} & \xrightarrow{E} & \mathbb{C}/\Lambda \\
S & \downarrow & S & \downarrow & E & \downarrow & E \\
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \mathcal{R} & \xrightarrow{E} & \mathbb{C}/\Lambda \\
\end{array}
\]

$\phi$ measure conjugation.

We can generalize what shown for the family of substitutions

\[\sigma_t : 1 \mapsto 1^{t+1}2, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 1^t5, \ 5 \mapsto 1\]

$\rightarrow (X_\sigma, S, \mu)$ is the first return of a toral translation.
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- **Projecting well** → projection of patches onto $E^s$ behaves well.
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- Roots of the **neutral polynomial** of modulus one → measure disjointness in the set equation.

- **Positivity**: $\bigwedge_{i=1}^{k} M_\sigma$ can have negative entries. Can we control cancellation? Can we control it using orientation of faces? For Tribo:

  $$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Possible definition of positivity: $|M_k|^j = |M_k^j|$, for all $j \in \mathbb{N}$. 
Guiding philosophy: try to turn the substitution into an irreducible one!
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Figure: Changing suitably the projection we get different polygonal tilings by some faces of three different types.
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- Pisot conjecture for reducible substitutions?
Strange examples

(Joint with X. Bressaud, T. Jolivet)

The geometric interpretation seems to get harder for other substitutions, not satisfying the strong coincidence condition:

\[ \sigma : 1 \mapsto 213, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 21 \]

\[ \text{char}(M_\sigma) = (x^2 + x + 1)(x^3 - 2x^2 + x - 1) \]
Lifting in the neutral space

Projection $\pi_{s,n} : \mathbb{R}^d \rightarrow E^s \oplus E^n$. 

[Diagram showing a projection from $\mathbb{R}^d$ to $E^s \oplus E^n$]
Lifting in the neutral space

Projection $\pi_{s,n} : \mathbb{R}^d \to E^s \oplus E^n$.

Criterion to know whether we get finitely many layers and NEW strong coincidence condition.
Gluing together

Projecting down suitably we can glue the subtiles together...

Figure: Symbolic splitting associated with the irreducible substitution \( \tau : 1 \mapsto 12, 2 \mapsto 32, 3 \mapsto 1 \).

... and obtain the connection with an irreducible substitution.

Philosophy: dynamically the reducible substitutive system behaves exactly as the irreducible one, after identifying some letters / changing projection. Technique: symbolic splitting.
Figure: Rauzy fractal of the Hokkaido substitution in $E^s \oplus E^n$. The points distribute with logarithmic growth on a two-dimensional lattice.
A self-induced IET substitution

\[ \sigma : 1 \mapsto 124, \ 2 \mapsto 1224, \ 3 \mapsto 124334, \ 4 \mapsto 12434 \]

\[ \text{char}(M_\sigma) = x^4 - 7x^3 + 13x^2 - 7x + 1, \ \beta_1, \beta_2 > 1, \ \beta_3, \beta_4 < 1 \]

Geometric representation → two fractal windows generated by

\[
E_2(\sigma)(x, a) = \sum_{a \to b} (M_\sigma x + l(p), b)
\]

\[
E_2^*(\sigma)(x, a)^* = \sum_{b \to a} (M_\sigma^{-1}(x - l(p)), b)^*
\]

Argue with similar hypotheses as for the reducible case: positivity, projecting-well, geometric finiteness property, etc.

Very complicated to state results in full generality! See Sage...