Quasicrystal Cooling

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TransTile
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Quasicrystals

Penrose quasiperiodic tiling vs $\text{Al}_{71}\text{Mn}_{24}\text{Fe}_{5}$ quasicrystal:
Structure & Growth

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**Question 1**
Which (aperiodic) tilings can be characterized by finite patterns?
Structure & Growth

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![Allowed Patterns](image)

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![Forbidden Patterns](image)

**Question 1**
Which (aperiodic) tilings can be characterized by finite patterns?

**Question 2**
How to “grow” a tiling from finite patterns which characterize it?
Deceptions and correcting flips

Let us try to grow tile by tile a Penrose tiling.
We simply need to form only allowed patterns.
Deceptions and correcting flips

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Deceptions and correcting flips

Oops! We created *deceptions*! Usual business with aperiodicity…
Deceptions and correcting flips

Let us forget allowed patterns and just grow a “random” tiling.
Deceptions and correcting flips

We shall try to correct it afterward by local moves called flips.
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Deceptions and correcting flips

Do we eventually get only allowed patterns? When?
Formally

Consider the tilings by rhombi of a finite simply connected region. Fix a finite set of forbidden finite patterns. Define the energy $E$ of a tiling as its number of forbidden patterns.

Consider the discrete time Markov chain at temperature $T$:

- pick a vertex \textit{u.a.r.}
- choose a flip type \textit{u.a.r.}
- if possible, perform the flip with probability $\exp(-\Delta E/T)$
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Consider the tilings by rhombi of a finite simply connected region.
Fix a finite set of forbidden finite patterns.
Define the *energy* $E$ of a tiling as its number of forbidden patterns.
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- if possible, perform the flip with probability $\exp(-\Delta E/T)$

The chain is known to be ergodic at $T > 0$ (Kenyon'93).
Is it *rapidly mixing*? For any $T$ or with some *phase transition*?
Example 1: dimers at infinite temperature

At $T = \infty$, flips are all performed with probability 1. Forbidden patterns (energy) do not play any role!

This is nothing but the usual random sampling of dimer tilings. Ask Dana or Benoît for the mixing time, Cédric for the limit shape!
Example 2: alternating dimers at zero temperature

Forbid adjacent identical tiles. Forbid flip increasing the energy.
Example 2: alternating dimers at zero temperature

It characterize the tilings whose dimers alternate (groundstate).
Example 2: alternating dimers at zero temperature

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Back to quasicrystals

A similar process on Penrose tiling seems to rapidly converge.
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