Markov Random Fields and Gibbs States

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Outline

- Homomorphism spaces
- Markov random fields and Gibbs states
- When are Markov random fields Gibbs states?
- Describing conditions on the support
  - All Markov random fields are Gibbs: Dismantlable graphs and the 3-coloured chessboard
  - Not all Markov random fields are Gibbs: The square island shift
- The pivot property
Some Notation and Setting

$G = (V_G, E_G)$ is a locally-finite undirected graph.

$A$ is a finite set of symbols.

$X \subset A$.

$V_G$ is a closed set.

For a finite set $A \subset V_G$ and a pattern $a : A \rightarrow A$, $\[a\]_A = \{x \in X | x|_A = a\}$ (Cylinder set).

$\partial A = \{v \in V_G \setminus A | v \sim w \in A\}$ (Boundary).
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\end{itemize}

- Elements of $A$
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  [a]_A := \{ x \in X \mid x|_A = a \} \quad \text{(Cylinder set)}
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Elements of \( A \)

Cylinder set \([4,3,1]_A\)
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Homomorphism Spaces

$H = (V_H, E_H)$ is a finite undirected graph without multiple edges.

$X = \text{Hom}(Z^d, H)$ is the space of all graph homomorphisms from $Z^d$ to $H$.

Examples: (The 3-coloured chessboard)
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Examples: (Hard square model)
Safe Symbol
Safe Symbol

$X$ has a safe symbol $\star$ if for all $x \in X$ and $n \in \mathcal{V}_G$, the configuration $y$ given by

$$y_m = \begin{cases} x_m & \text{if } m \neq n \\ \star & \text{if } m = n \end{cases}$$

is an element of $X$. 
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The space $X = \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has a safe symbol $\star$ if and only if for all $v \in \mathcal{H}$, $\star \sim v$. 
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The space $X = Hom(\mathbb{Z}^d, \mathcal{H})$ has a safe symbol $\star$ if and only if for all $v \in \mathcal{H}$, $\star \sim v$. Thus 0 is a safe symbol for the hard square model but the 3-coloured chessboard model does not have any safe symbol.
A Markov random field (MRF) is a probability measure $\mu$ on $A^\nu_g$.

Markov Random Fields
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A Markov random field (MRF) is a probability measure $\mu$ on $\mathcal{A}^\mathcal{V}_g$ such that for all finite $A, B \subset \mathcal{V}_g$, $\partial A \subset B \subset A^c$.

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- Elements of $B$
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$$\mu([a]_A \bigg| [b]_B) = \mu([a]_A \bigg| [b]_{\partial A}).$$

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The set of conditional measures $\mu([\cdot]_A \mid [b]_{\partial A})$ for all $A \subset \mathcal{V}_G$ finite and $b \in \mathcal{A}^{\partial A}$ is called the specification for the measure $\mu$. It might not have any finite description.
Gibbs States

A nearest neighbour (n.n.) interaction on $X$. 

If $G = \mathbb{Z}^d$ the specification of a Gibbs state with a shift-invariant n.n. interaction has a finite description: all we need is the interaction $V$. 

Gibbs States

A nearest neighbour (n.n.) interaction on $X$ is a function

$$V : \{ [a]_A \mid A \text{ is an edge or vertex in } \mathcal{G} \} \rightarrow \mathbb{R}.$$
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where $Z_{A,x|\partial A}$ is the uniquely determined normalising factor dependent upon $A$ and $x|\partial A$. 
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A Gibbs state with a n.n. interaction \( V \) is a Markov random field \( \mu \) such that for all \( x \in \text{supp}(\mu) \) and finite set \( A \subset \mathcal{V}_G \)

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**Example:** If a shift-invariant n.n. interaction on the hard square model is given by

![Graph H](image)

![Interaction V](image)
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\begin{align*}
&\text{Graph } H \\
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that is,

\[
V([00]_0,\vec{e}_i) = V([10]_0,\vec{e}_i) = V([01]_0,\vec{e}_i) = 0,
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![Graph H and Interaction V](image)

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then

\[ \mu([x]_A \mid [x]_{\partial A}) = \frac{\prod_{C \subseteq A \cup \partial A} e^{V([x]_C)}}{Z_{A,x|_{\partial A}}} = \frac{e^{\text{number of 1's in } x|_{A\cup\partial A}}}{Z_{A,x|_{\partial A}}}. \]
Question: Under what conditions on the support is every MRF a Gibbs state for some n.n. interaction?
Positive Results

Conditions on the support such that every MRF is Gibbs for some n.n. interaction

The support has a safe symbol: Hammersley and Clifford ('71)

Algebraic conditions on the support and $G$ is a finite graph: Sturmfels, Gieger and Meek ('06)

Conditions on the graph $G$: Lauritzen ('96)

For shift-invariant measures and $G = \mathbb{Z}$ under some mixing conditions on the support (but infinite set of symbols): Georgii ('88)

For shift-invariant measures and $G = \mathbb{Z}$: Chandgotia, Han, Marcus, Meyerovitch and Pavlov ('11)

New Results:
The support is the 3-coloured chessboard model. A generalisation of the Hammersley-Clifford theorem when $G$ is bipartite.
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Counterexamples

MRFs which need not be Gibbs for any n.n. interaction:

When $G$ is a finite graph: Moussouris (’74)

When $G = \mathbb{Z}$ and the measure is not shift-invariant: Dobrushin (’68)

When the alphabet is countable: Georgii (’88)

New Results:

For $G = \mathbb{Z}^2$ we constructed a family of shift-invariant MRFs
which are not Gibbs for any shift-invariant finite-range interaction (not just nearest neighbour).
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Dismantlable Graphs

Consider an undirected finite graph $\mathcal{H}$.

\[
\begin{array}{c}
\text{w} \\
\text{u} \quad \text{t} \\
\text{v}
\end{array}
\]
Dismantlable Graphs

Consider an undirected finite graph $\mathcal{H}$. $N(v)$ denotes the *neighbourhood* of $v$ in $\mathcal{H}$, that is,

$$N(v) = \{ s \in \mathcal{H} \mid s \sim v \}.$$
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$\mathcal{H}$ can be folded to a graph $\mathcal{H} \setminus \{ v \}$ if there exists a vertex $w \in \mathcal{H}$ such that $N(v) \subset N(w)$.

![Graph Diagram](image-url)
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Consider an undirected finite graph $H$. $N(v)$ denotes the *neighbourhood* of $v$ in $H$, that is,

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Dismantlable Graphs

If $\text{Hom}(\mathbb{Z}_d, H)$ has a safe symbol $\star$, then for all vertices $v \in V_H$, $N(v) \subseteq N(\star) = V_H$ and thus all vertices $v$ can be folded into $\star$. Then $H$ is dismantlable.

And there are graphs where no folding is possible. Let $C_n$ denote the $n$-cycle.
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Dismantlable Graphs

If $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has a safe symbol $\star$ then for all vertices $v \in \mathcal{V}_H$, $\mathcal{N}(v) \subset \mathcal{N}(\star) = \mathcal{V}_H$ and thus all vertices $v$ can be folded into $\star$. Then $\mathcal{H}$ is dismantlable.

However there are dismantlable graphs $\mathcal{H}$ even if $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ does not have a safe symbol.
Dismantlable Graphs

If $Hom(\mathbb{Z}^d, \mathcal{H})$ has a safe symbol $\star$ then for all vertices $v \in V_{\mathcal{H}}$, $N(v) \subseteq N(\star) = V_{\mathcal{H}}$ and thus all vertices $v$ can be folded into $\star$. Then $\mathcal{H}$ is dismantlable.

And there are graphs where no folding is possible. Let $C_n$ denote the $n$-cycle.
New Results

Theorem (Chandgotia and Meyerovitch '13, Chandgotia '14)
If $H$ is either $C^n$ for some $n$ or dismantlable then any MRF on $\text{Hom}(\mathbb{Z}^d, H)$ is a Gibbs state for some $n.n.$ interaction. Further if the MRF is shift-invariant then it is a Gibbs state for some shift-invariant $n.n.$ interaction. In fact we prove further and generalise the Hammersley-Clifford theorem when the underlying graph $G$ is bipartite. How can such a theorem be proved?
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How can such a theorem be proved?
Pivot Property

A space $X$ is said to satisfy the pivot property if for all $x, y \in X$ which differ only on finitely many sites there exists a chain $x = x^1, x^2, x^3, \ldots, x^n = y \in X$ such that $x^i, x^{i+1}$ differ on at most a single site.
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A space $X$ is said to satisfy the **generalised pivot property** if there exists $K > 0$ such that for all $x, y \in X$ which differ only on finitely many sites there exists a chain $x = x^1, x^2, x^3, \ldots, x^n = y \in X$ such that $x^i, x^{i+1}$ differ only on a region of diameter at most $K$.

Examples:
- $\text{Hom}(\mathbb{Z}^d, H)$ when $H$ is dismantlable.
- $\text{Hom}(\mathbb{Z}^d, H)$ when $H$ does not have a four-cycle.
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The 3-coloured Chessboard
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The 3-coloured chessboard has the pivot property.

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1 & 0 & 1 & 0 & 1 & 0 & 1 \\
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\end{array}
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The 3-coloured Chessboard

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Suppose $\mu$ is a Markov random field whose support has the pivot property.
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$\mu(\{x\}_{0} \cup \partial 0) \mu(\{y\}_{0} \cup \partial 0)$ for configurations $x, y$ which differ only at $0$, the origin. Thus the space of specifications on $\text{supp}(\mu)$ can be parametrised by finitely many parameters.
Suppose $\mu$ is a Markov random field whose support has the pivot property. Then given $x, y \in \text{supp}(\mu)$ that differ exactly on $F$ there exists a chain $x = x^1, x^2, \ldots, x^n = y$ where $x^i, x^{i+1}$ differ exactly at a site $m_i \in \mathbb{Z}^2$ and consequently

$$
\frac{\mu([x]_F | [x]_{\partial F})}{\mu([y]_F | [x]_{\partial F})} = \prod_{i=1}^{n-1} \frac{\mu([x^i]_F | [x^i]_{\partial F})}{\mu([x^{i+1}]_F | [x^i]_{\partial F})}
$$

$$
= \prod_{i=1}^{n-1} \frac{\mu([x^i]_{m_i} | [x^i]_{\partial m_i})}{\mu([x^{i+1}]_{m_i} | [x^i]_{\partial m_i})}.
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Therefore the entire specification is determined by finitely many parameters viz. $\frac{\mu([x]_{0 \cup \partial 0})}{\mu([y]_{0 \cup \partial 0})}$ for configurations $x, y$ which differ only at 0, the origin.
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Thus the space of specifications on $\text{supp}(\mu)$ can be parametrised by finitely many parameters.
Question: Suppose we are given a nearest neighbour shift of finite type with the pivot property. Is there an algorithm to determine the number of parameters which describes the specification?
A specification supported on the 3-coloured chessboard is determined the quantities $v_1 = \frac{\mu(\begin{bmatrix} 1 & 0 & 1 \\ 1 \\ 1 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 2 & 1 \\ 1 \\ 1 \end{bmatrix})}$, $v_2 = \frac{\mu(\begin{bmatrix} 2 & 2 & 1 \\ 2 \\ 1 \end{bmatrix})}{\mu(\begin{bmatrix} 2 & 0 & 2 \\ 2 \\ 1 \end{bmatrix})}$, and $v_3 = \frac{\mu(\begin{bmatrix} 0 & 0 & 2 \\ 0 \\ 0 \end{bmatrix})}{\mu(\begin{bmatrix} 0 & 1 & 0 \\ 0 \\ 0 \end{bmatrix})}$. If $\mu$ is a Gibbs measure with nearest neighbour interaction $V$ then $v_1 = \exp\left(V(01) + V(10) + V(01) + V(01) - V(21) - V(12) - V(21) - V(12)\right)$, $v_2 = \exp\left(V(12) + V(21) + V(21) + V(12) - V(02) - V(20) - V(02) - V(20)\right)$, and $v_3 = \exp\left(V(02) + V(20) + V(20) + V(02) - V(01) - V(10) - V(01) - V(10)\right)$. $\mu$ is Gibbs if and only if $v_1 v_2 v_3 = 1$. 
A specification supported on the 3-coloured chessboard is determined the quantities $v_1 = \frac{\mu(\begin{bmatrix} 1 & 0 & 1 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix})}$, $v_2 = \frac{\mu(\begin{bmatrix} 2 & 1 & 2 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})}$, $v_3 = \frac{\mu(\begin{bmatrix} 2 & 0 & 2 \end{bmatrix})}{\mu(\begin{bmatrix} 0 & 2 & 2 \end{bmatrix})}$.

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\[ v_1 = \mu \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array} \right), \quad v_2 = \mu \left( \begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array} \right), \quad v_3 = \mu \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right) \].

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determined the quantities \( v_1 = \frac{\mu\left( \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \right)}{\mu\left( \begin{bmatrix} 1 & 1 \end{bmatrix} \right)} \), \( v_2 = \frac{\mu\left( \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \right)}{\mu\left( \begin{bmatrix} 1 & 0 \end{bmatrix} \right)} \) and
\( v_3 = \frac{\mu\left( \begin{bmatrix} 0 & 2 & 0 \end{bmatrix} \right)}{\mu\left( \begin{bmatrix} 0 & 1 \end{bmatrix} \right)} \).

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A specification supported on the 3-coloured chessboard is determined the quantities \( v_1 = \frac{\mu(\begin{bmatrix} 1 & 0 & 1 \end{bmatrix})}{\mu(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})} \), \( v_2 = \frac{\mu(\begin{bmatrix} 2 & 1 & 2 \end{bmatrix})}{\mu(\begin{bmatrix} 2 & 0 & 2 \end{bmatrix})} \) and \( v_3 = \frac{\mu(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 \end{bmatrix})}{\mu(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 \end{bmatrix})} \). If \( \mu \) is a Gibbs measure with nearest neighbour interaction \( V \) then

\[
\begin{align*}
v_1 &= \exp(V(01) + V(10) + V(0_1) + V(0_1) \\
&\quad - V(21) - V(12) - V(2_1) - V(1_2)), \\
v_2 &= \exp(V(12) + V(21) + V(2_1) + V(1_2) \\
&\quad - V(02) - V(20) - V(2_0) - V(2_0)), \\
v_3 &= \exp(V(02) + V(20) + V(2_0) + V(0_2) \\
&\quad - V(01) - V(10) - V(0_1) - V(1_0)).
\end{align*}
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Square Island Shift

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There are two kinds of squares: ones with red dots and ones without red dots which float in a sea of blanks.
New Results

There are infinitely many independent parameters required to describe a specification of a shift-invariant MRF on the square island shift, for instance, the ratios of the probability of a big square with red dots and the probability of a square of the same size without red dots.

It does not have the generalised pivot property

Theorem (Chandgotia and Meyerovitch '13)

There exists a shift-invariant MRF supported on the square island shift which is not Gibbs for any shift-invariant finite-range interaction.

Is there a more natural example?
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**Question:** If $\text{Hom}(\mathbb{Z}^2, \mathcal{H})$ has the generalised pivot property, can you determine the minimum number of parameters required to determine the specification of a Markov random field?
Thank You!