Embedding computations in tilings
(a perspective of the course)

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30 May 2016
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**Tiling:** a mapping $f: \mathbb{Z}^2 \rightarrow \tau$

that respects the matching rules
A shift of finite type (SFT):

- A finite set of letters \( \tau \)
- A finite set of forbidden (finite) patterns \( F \)
- SFT: the set of all configurations \( f: \mathbb{Z}^2 \to \tau \) that does not contain forbidden patterns

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\( T \in \mathbb{Z}^2 \) is a **period** if \( f(x + T) = f(x) \) for all \( x \).
super-classic facts:

▶ SFT $\leadsto$ tilings

▶ if you can tile arbitrarily large square, than you can tile the infinite plane (compacteness)

▶ if there exists a $\tau$-tiling with one period $T$, then there exists another tiling with two non collinear periods $T_1, T_2$

▶ there exist tile sets $\tau$ s.t. all $\tau$-tilings are aperiodic

▶ there exists a tile set $\tau$ s.t. all $\tau$-tilings are non-computable

▶ given a tile set $\tau$ we cannot algorithmically decide whether there exists a $\tau$-tiling of $\mathbb{Z}^2$
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▶ any effective (polynomial time) real-life algorithm can be performed by a Turing machine in polynomial time
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Two techniques of embedding a computation in a tiling:
- From self-referential programs to self-similar tilings (goes back to J. von Neumann)
- From arithmetic in Sturmian numeration system to tilings (J. Kari)

Very standard application:
- A construction of an aperiodic tile set

Less standard application:
- Aperiodicity + quasiperiodicity (and even minimality)
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- Some applications of the self-simulating tilings:
  - The tiling problem is undecidable [Berger 1966]
  - A tile set with only non-computable tilings [Hanf & Myers 1974]
  - A tile set with highly aperiodic tilings [?]
  - Robust (error-correcting) tilings [?]
  - An effective shift is isomorphic to a subaction of a sofic shift [Hochman 2009, Aubrun & Sablik 2013]
  - A minimal effective shift can be simulated by a minimal SFT [?]

- Another remarkable result:
  - Kari’s technique gives non self-similar tilings [T. Monteil]
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Come to the lectures!