Embedding computations in tilings
(Part 1: fixed point tilings)

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What is a **tile**?
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In this mini-course:

**Color:** an element of a finite set $C = \{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot\}$
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i.e, an element of $C^4$, e.g., \[ \text{\[ ]} \]
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**Tile set:** a set $\tau \subset C^4$
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**Tiling**: a mapping $f : \mathbb{Z}^2 \rightarrow \tau$
that respects the matching rules
Tiling: a mapping $f : \mathbb{Z}^2 \to \tau$ such that

$$f(i,j).\text{right} = f(i+1,j).\text{left}, \quad \text{e.g., } \begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & \cdot \end{array} + \begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & \cdot \end{array}$$

$$f(i,j).\text{top} = f(i,j+1).\text{bottom}, \quad \text{e.g., } \begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & \cdot \end{array} + \begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & \cdot \end{array}$$
Tiling: a mapping $f : \mathbb{Z}^2 \rightarrow \tau$ such that

$$f(i, j).\text{right} = f(i + 1, j).\text{left}, \quad \text{e.g.,} \quad \square + \square$$

$$f(i, j).\text{top} = f(i, j + 1).\text{bottom}, \quad \text{e.g.,} \quad \quad +$$

Example. A finite pattern from a valid tiling:
τ-tiling is a mapping $f : \mathbb{Z}^2 \rightarrow \tau$ that respects the local rules.
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\( T \in \mathbb{Z}^2 \) is a **period** if \( f(x + T) = f(x) \) for all \( x \).
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\( T \in \mathbb{Z}^2 \) is a **period** if \( f(x + T) = f(x) \) for all \( x \).

**Theorem.** There exists a tile set \( \tau \) such that
(i) \( \tau \)-tilings exist, and
τ-tiling is a mapping $f : \mathbb{Z}^2 \to \tau$ that respects the local rules.

$T \in \mathbb{Z}^2$ is a **period** if $f(x + T) = f(x)$ for all $x$.

**Theorem.** There exists a tile set $\tau$ such that

(i) $\tau$-tilings exist, and

(ii) all $\tau$-tilings are aperiodic.
A construction of an aperiodic tile set:
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- define **self-similar** tile sets
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- define **self-similar** tile sets
- observe that *every* self-similar tile set is aperiodic
A construction of an aperiodic tile set:

- define **self-similar** tile sets
- observe that *every* self-similar tile set is aperiodic
- construct *some* self-similar tile set
an $N \times N$ square made of matching $\tau$-tiles
Fix a tile set \( \tau \) and an integer \( N > 1 \).
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**Definition 1.** A $\tau$-macro-tile: an $N \times N$ square made of matching $\tau$-tiles.
Fix a tile set $\tau$ and an integer $N > 1$.

**Definition 1.** A $\tau$-macro-tile: an $N \times N$ square made of matching $\tau$-tiles.

**Definition 2.** A tile set $\rho$ is simulated by $\tau$: there exists a family of $\tau$-macro-tiles $R$ such that

- $R$ is isomorphic to $\rho$, and
- every $\tau$-tiling can be *uniquely* split by an $N \times N$ grid into macro-tiles from $R$. 
Example.

A tile set $\rho$: Trivial tile set (only one color)
Example.

A tile set $\rho$: Trivial tile set (only one color)
A tile set $\tau$: A tile set that simulates a trivial tile set $\rho$
Example.

A tile set \( \rho \): Trivial tile set (only one color)
A tile set \( \tau \): A tile set that simulates a trivial tile set \( \rho \)

\[
\begin{array}{c}
(i, j + 1) \\
(i, j) \\
(i + 1, j) \\
(i, j)
\end{array}
\]
Self-similar tile set: a tile set that simulates a set of macrotiles isomorphic to itself.
**Self-similar** tile set: a tile set that simulates a set of macrotiles *isomorphic* to itself.

**Proposition.** Self-similar tile set is aperiodic.
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**Proposition.** Self-similar tile set is aperiodic.

Sketch of the proof:
Self-similar tile set: a tile set that simulates a set of macrotiles isomorphic to itself.

**Proposition.** Self-similar tile set is aperiodic.

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Self-similar tile set: a tile set that simulates a set of macrotiles isomorphic to itself.

Proposition. Self-similar tile set is aperiodic.
Sketch of the proof:
Simulating a given tile set \( \rho \) by macro-tiles.
Simulating a given tile set $\rho$ by macro-tiles.
Representation of the tile set $\rho$: 

\[ \begin{aligned}
\text{colors of a tile set } \rho &\Rightarrow k\text{-bits strings} \\
\text{a tile set } \rho &\Rightarrow \text{a predicate } P(x_1, x_2, x_3, x_4) \\
\text{on } 4\text{-tuples of colors} &
\end{aligned} \]
Simulating a given tile set $\rho$ by macro-tiles.

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- colors of a tile set $\rho \implies k$-bits strings
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- colors of a tile set $\rho$ $\implies$ $k$-bits strings
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Simulating a given tile set $\rho$ by macro-tiles.

Representation of the tile set $\rho$:

- Colors of a tile set $\rho \implies k$-bits strings
- A tile set $\rho \implies$ a predicate $P(x_1, x_2, x_3, x_4)$ on 4-tuples of colors
- A TM that accepts only 4-tuples of colors for the $\rho$-tiles
The scheme of implementation:
A more generic construction:
universal TM + program
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universal TM + program

A fixed point: simulating tile set = simulated tile set
How to get **aperiodicity + quasiperiodicity**?
How to get aperiodicity + quasiperiodicity?

The problematic part is the computation zone...
Duplicate all $2 \times 2$ patterns that may appear in the computation zone!
A slot for a $2 \times 2$ patterns from the computation zone:

<table>
<thead>
<tr>
<th></th>
<th>$(i, j + 4)$</th>
<th>$(i + 1, j + 4)$</th>
<th>$(i + 2, j + 4)$</th>
<th>$(i + 3, j + 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(i, j + 3)$</td>
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<tr>
<td>$(i, j)$</td>
<td>$(s, t + 2)$</td>
<td>$(s + 1, t + 2)$</td>
<td>$(i + 3, j + 3)$</td>
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