A counterexample to a conjecture of Lagarias and Pleasants

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1 Introduction

In [1], Lagarias and Pleasants propose the following conjecture:

**Conjecture 1** Any aperiodic Delone set \( X \) in \( \mathbb{R}^d \) satisfies

\[
\limsup_{T \to \infty} \frac{N_X(T)}{T^d} > 0
\]

where \( N_X(T) \) is the patch-counting function of \( X \), that is, the number of patches of radius \( T \) in \( X \) up to translation.

In the setting of multidimensional words, the above conjecture translates as follows:

**Conjecture 2** Let \( A \) be an alphabet and \( u \) a word in \( A^{2^d} \), where \( d \geq 1 \). Let \( P(n) \) be the one-parameter complexity function of \( u \), that is, the number of factors of size \( n \times \cdots \times n \) that occur in \( u \). If \( P(n) = o(n^d) \), then \( u \) is periodic.

Conjecture 2 is true for \( d = 1 \) and \( d = 2 \). Actually, stronger statements hold. For \( d = 1 \), the hypothesis \( P(n) = o(n^d) \) can be replaced with \( \exists n, P(n) \leq n \), by the theorem of Morse and Hedlund [2]. For \( d = 2 \), it can be replaced with \( \exists n, P(n) \leq n^2/16 \) by the result of Quas and Zamboni [4], and it is conjectured that it can be replaced with \( \exists n, P(n) \leq n^2 \) (Nivat’s conjecture [3]). However, Sander and Tijdeman [5] showed that the conjecture cannot be strengthened in a similar way for \( d \geq 3 \).

In this note, we construct a counterexample to Conjecture 2 (and thus also to Conjecture 1) when \( d \geq 3 \), showing that \( P(n) \) can be as low as \( n^2/o(1) \) for any \( d \). Moreover, the multidimensional word we construct is uniformly recurrent, so the conjectures cannot be repaired by just adding this condition.
2 Construction

Let $d \geq 3$. We construct a $d$-dimensional word $u \in \{0, 1\}^Z$.

Let $(r_i)$ be an increasing sequence of positive integers, to be chosen later, with $r_0 = 1$, and let 

$$q_m = \prod_{i=0}^{m-1} r_i,$$

so that $q_0 = q_1 = 1$.

We define $u = (u_x)_{x \in \mathbb{Z}^d}$, $x = (x_1, x_2, \ldots, x_d)$, as follows:

- If $x_d = 0$, then $u_x = 0$.
- If $x_d \neq 0$, let $m$ be the largest integer such that $q_m$ divides $x_d$ (note that $m \geq 1$), and let $j \in \{1, \ldots, d-1\}$ be such that $j = m \pmod{d-1}$. Then $u_x = 1$ if $q_m-1$ divides $x_j$, $u_x = 0$ otherwise.

Theorem 1 The $d$-dimensional word $u$ defined above is aperiodic and uniformly recurrent, and its one-parameter complexity function satisfies $P(n) = o(n^3)$.

The remaining sections are devoted to the proof of this theorem.

3 Aperiodicity

Let us first check that the $d$-dimensional word $u$ is aperiodic.

Indeed, let $t = (t_1, t_2, \ldots, t_d) \in \mathbb{Z}^d \setminus \{0\}$.

If $t_d \neq 0$, consider $x = (0, \ldots, 0, -t_d)$ and $y = (t_1, \ldots, t_d-1, 0)$. By construction, $u_x = 1$ and $u_y = 0$, so $t = y - x$ cannot be a period.

If $t_d = 0$, choose $j$ such that $t_j \neq 0$ and $m$ such that $m = j \pmod{d-1}$ and $q_{m-1} > |t_j|$. Consider then $x = (0, \ldots, 0, q_m)$ and $y = (t_1, \ldots, t_{d-1}, q_m)$.

By construction, $u_x = 1$ and $u_y = 0$, so $t = y - x$ cannot be a period either.

We have just proved that $u$ does not have any non-trivial period.

4 Uniform recurrence

To prove that $u$ is uniformly recurrent, it is sufficient to show that for infinitely many $n$, there is a $N$ such that the factor of size $n \times \cdots \times n$ centered at 0 occurs in any factor of size $N \times \cdots \times N$. We shall prove this when $n$ is of the form $2q_p - 1$ for some $p \geq 1$, with $N = q_{p+2} + 2q_p - 2$. Actually, we prove that the factor of size $(2q_p - 1) \times \cdots \times (2q_p - 1)$ centered at 0 also occurs centered at $(k_1q_{p+1} + q_p, \ldots, k_{d-1}q_{p+1} + q_p, kdq_{p+2})$ for any $(k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$.

Indeed, let $x$ be such that $|x_i| < q_p$ for all $i$, and $y = x + (k_1q_{p+1} + q_p, \ldots, k_{d-1}q_{p+1} + q_p, kdq_{p+2})$.

If $x_d = 0$, then $u_x = 0$, and $q_{p+2}$ divides $y_d$ but $q_{p+1}$ does not divide $y_j$ for any $j < d$, so that $u_y = 0 = u_x$. 

2
If \( x_d \neq 0 \), let \( m \) be the largest integer such that \( q_m \) divides \( x_d \). Note that
\[ 1 \leq m < p, \text{ and that } m \text{ is also the largest integer such that } q_m \text{ divides } y_d. \]
Let \( j \in \{1, \ldots, d-1\} \) be such that \( j = m \pmod{d-1} \). Then \( u_x = 1 \) if and only if \( q_{m-1} \) divides \( x_j \), and \( u_x = 1 \) if and only if \( q_{m-1} \) divides \( y_j \). As \( y_j \equiv x_j \pmod{q_{m-1}} \), it follows that \( u_y = u_x \).

## 5 Complexity

Fix \( n \geq 1 \). Let \( p \) be the largest integer such that \( n \geq q_p \), and \( v \) be an \( n \times \cdots \times n \) factor of \( u \), occurring at some position \( x \).

For each \( z \) such that \( 0 \leq z < n \), let \( v_z \) be the \( z \)-th layer of \( v \), i.e., the \( d-1 \)-dimensional word obtained from \( v \) by fixing the last coordinate to be \( z \).

By construction of \( u \), each layer \( v_z \) is determined by the value of \( x_j \) modulo \( q_{m-1} \), where \( m = m(z) \) is the largest integer such that \( q_m \) divides \( x_d + z \) and \( j = m \pmod{d-1} \).

Order the values taken by \( m(z) \) in decreasing order: \( m_0, m_1, \ldots, m_s \). For \( 0 \leq i \leq s \), let \( j_i = m_i \pmod{d-1} \) and choose some \( z_i \) such that \( m(z_i) = m_i \).

Observe that \( m_0 \) may be arbitrarily large, but \( m_1 \leq p \) since \( q_{m_1} \) divides \( z_1 - z_0 \), so that \( q_{m_1} < n \). Also, once \( z_0 \) is fixed, then \( m(z) \) is determined for all \( z \neq 0 \): it is the largest \( m \) such that \( q_m \) divides \( z - z_0 \). Observe also that \( q_{m_0+1} > n \geq q_p \), so that \( m_0 \geq p \).

We distinguish two cases.

First case: \( m_0 - 1 \leq p \). Then \( m_0 \) has only two possible values, \( p \) and \( p + 1 \). The factor \( v \) is entirely defined by \( m_0, z_0 \), and \( x_j \pmod{q_{m_i-1}} \) for \( 0 \leq i \leq s \).

There are at most \( 2n \prod_{i=0}^{p-1} q_i \) such factors.

Second case: \( m_0 - 1 > p \). Then either layer \( v_{z_0} \) is non-zero, and it is determined by \( j_0 \) and some \( 0 \leq y < n \) such that \( x_{j_0} + y = 0 \pmod{q_{m_0-1}} \), or layer \( v_{z_0} \) is zero, and we set \( y = n \). The factor \( v \) is entirely defined by \( j_0, z_0, y, \) and \( x_j \pmod{q_{m_i-1}} \) for \( 1 \leq i \leq s \). As \( m_1 \leq p \), there are at most \( (d-1)n(n+1) \prod_{i=0}^{p-1} q_i \) such factors.

Summing both numbers, we find that

\[
P(n) \leq n(2q_p + (d-1)(n+1)) \prod_{i=0}^{p-1} q_i \leq 2dn^2 \prod_{i=0}^{p-1} q_i.
\]

We can now choose the sequence \( (r_i) \). Let \( (A_i) \) be any nondecreasing sequence of integers tending to \( +\infty \) (as slowly as we want). We define \( (r_i) \) inductively, starting with \( r_0 = 1 \). Assume \( r_i \) is constructed for \( i < m \), so that \( q_j = \prod_{i=0}^{j_1} r_i \) is defined for \( j \leq m \). Let then \( r_m \) be the smallest integer such that \( r_m > r_{m-1} \) and \( A_{q_m r_m} \geq 2d \prod_{i=0}^{m} q_i \).

We then have \( P(n) \leq n^2 A_{q_p} \leq n^2 A_n \) for all \( n \geq q_2 \). Taking \( A_i = [\sqrt{i}] \) for instance, we get \( P(n) = o(n^3) \).
References


