A counterexample to a conjecture of Lagarias and Pleasants

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1 Introduction

In [1], Lagarias and Pleasants propose the following conjecture:

Conjecture 1 *Any aperiodic Delone set X in $\mathbb{R}^d$ satisfies*

$$\limsup_{T\to\infty} \frac{N_X(T)}{T^d} > 0$$

where $N_X(T)$ is the patch-counting function of X, that is, the number of patches of radius T in X up to translation.

In the setting of multidimensional words, the above conjecture translates as follows:

Conjecture 2 *Let A be an alphabet and $\mathbf{u}$ a word in $A^{\mathbb{Z}^d}$, where $d \geq 1$. Let $P(n)$ be the one-parameter complexity function of $\mathbf{u}$, that is, the number of factors of size $n \times \cdots \times n$ that occur in $\mathbf{u}$. If $P(n) = o(n^d)$, then $\mathbf{u}$ is periodic.*

Conjecture 2 is true for $d = 1$ and $d = 2$. Actually, stronger statements hold. For $d = 1$, the hypothesis $P(n) = o(n^d)$ can be replaced with $\exists n, P(n) \leq n$, by the theorem of Morse and Hedlund [2]. For $d = 2$. it can be replaced with $\exists n, P(n) \leq n^2/16$ by the result of Quas and Zamboni [4], and it is conjectured that it can be replaced with $\exists n, P(n) \leq n^2$ (Nivat's conjecture [3]). However, Sander and Tijdeman [5] showed that the conjecture cannot be strengthened in a similar way for $d \geq 3$.

In this note, we construct a counterexample to Conjecture 2 (and thus also to Conjecture 1) when $d \geq 3$, showing that $P(n)$ can be as low as $n^2/o(1)$ for any d. Moreover, the multidimensional word we construct is uniformly recurrent, so the conjectures cannot be repaired by just adding this condition.

2 Construction

Let $d \geq 3$. We construct a d-dimensional word $\mathbf{u} \in \{0,1\}^{\mathbb{Z}^d}$.

Let (r_i) be an increasing sequence of positive integers, to be chosen later, with $r_0 = 1$, and let

$$q_m = \prod_{i=0}^{m-1} r_i \ ,$$

so that $q_0 = q_1 = 1$.

We define $\mathbf{u} = (u_{\mathbf{x}})_{\mathbf{x} \in \mathbb{Z}^d}$, $\mathbf{x} = (x_1, x_2, \dots, x_d)$, as follows:

- If $x_d = 0$, then $u_{\mathbf{x}} = 0$.
- If $x_d \neq 0$, let m be the largest integer such that q_m divides x_d (note that $m \geq 1$), and let $j \in \{1, \dots, d-1\}$ be such that $j = m \pmod{d-1}$. Then $u_{\mathbf{x}} = 1$ if q_{m-1} divides x_j, $u_{\mathbf{x}} = 0$ otherwise.

Theorem 1 *The d-dimensional word $\mathbf{u}$ defined above is aperiodic and uniformly recurrent, and its one-parameter complexity function satisfies $P(n) = o(n^3)$.*

The remaining sections are devoted to the proof of this theorem.

3 Aperiodicity

Let us first check that the d-dimensional word $\mathbf{u}$ is aperiodic.

Indeed, let $\mathbf{t} = (t_1, t_2, \dots, t_d) \in \mathbb{Z}^d \setminus \{0\}$.

If $t_d \neq 0$, consider $\mathbf{x} = (0, \dots, 0, -t_d)$ and $\mathbf{y} = (t_1, \dots, t_{d-1}, 0)$. By construction, $u_{\mathbf{x}} = 1$ and $u_{\mathbf{y}} = 0$, so $\mathbf{t} = \mathbf{y} - \mathbf{x}$ cannot be a period.

If $t_d = 0$, choose j such that $t_j \neq 0$ and m such that $m = j \pmod{d-1}$ and $q_{m-1} > |t_j|$. Consider then $\mathbf{x} = (0, \dots, 0, q_m)$ and $\mathbf{y} = (t_1, \dots, t_{d-1}, q_m)$. By construction, $u_{\mathbf{x}} = 1$ and $u_{\mathbf{y}} = 0$, so $\mathbf{t} = \mathbf{y} - \mathbf{x}$ cannot be a period either.

We have just proved that $\mathbf{u}$ does not have any non-trivial period.

4 Uniform recurrence

To prove that $\mathbf{u}$ is uniformly recurrent, it is sufficient to show that for infinitely many n, there is a N such that the factor of size $n \times \cdots \times n$ centered at 0 occurs in any factor of size $N \times \cdots \times N$. We shall prove this when n is of the form $2q_p - 1$ for some $p \geq 1$, with $N = q_{p+2} + 2q_p - 2$. Actually, we prove that the factor of size $(2q_p - 1) \times \cdots \times (2q_p - 1)$ centered at 0 also occurs centered at $(k_1 q_{p+1} + q_p, \dots, k_{d-1} q_{p+1} + q_p, k_d q_{p+2})$ for any $(k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$.

Indeed, let $\mathbf{x}$ be such that $|x_i| < q_p$ for all i, and $\mathbf{y} = \mathbf{x} + (k_1 q_{p+1} + q_p, \dots, k_{d-1} q_{p+1} + q_p, k_d q_{p+2})$.

If $x_d = 0$, then $u_{\mathbf{x}} = 0$, and q_{p+2} divides y_d but q_{p+1} does not divide y_j for any $j < d$, so that $u_{\mathbf{y}} = 0 = u_{\mathbf{x}}$.

If $x_d \neq 0$, let m be the largest integer such that q_m divides x_d. Note that $1 \leq m < p$, and that m is also the largest integer such that q_m divides y_d. Let $j \in \{1, \ldots, d-1\}$ be such that $j = m \pmod{d-1}$. Then $u_{\mathbf{x}} = 1$ if and only if q_{m-1} divides x_j, and $u_{\mathbf{y}} = 1$ if and only if q_{m-1} divides y_j. As $y_j = x_j \pmod{q_{m-1}}$, it follows that $u_{\mathbf{y}} = u_{\mathbf{x}}$.

5 Complexity

Fix $n \geq 1$. Let p be the largest integer such that $n \geq q_p$, and v be an $n \times \cdots \times n$ factor of $\mathbf{u}$, occurring at some position $\mathbf{x}$.

For each z such that $0 \leq z < n$, let v_z be the z*-th layer* of v, i.e., the $d-1$-dimensional word obtained from v by fixing the last coordinate to be z. By construction of $\mathbf{u}$, each layer v_z is determined by the value of x_j modulo q_{m-1}, where $m = m(z)$ is the largest integer such that q_m divides $x_d + z$ and $j = m \pmod{d-1}$.

Order the values taken by $m(z)$ in decreasing order: m_0, m_1, ..., m_s. For $0 \leq i \leq s$, let $j_i = m_i \pmod{d-1}$ and choose some z_i such that $m(z_i) = m_i$. Observe that m_0 may be arbitrarily large, but $m_1 \leq p$ since q_{m_1} divides $z_1 - z_0$, so that $q_{m_1} < n$. Also, once z_0 is fixed, then $m(z)$ is determined for all $z \neq 0$: it is the largest m such that q_m divides $z - z_0$. Observe also that $q_{m_0+1} > n \geq q_p$, so that $m_0 \geq p$.

We distinguish two cases.

First case: $m_0 - 1 \leq p$. Then m_0 has only two possible values, p and $p+1$. The factor v is entirely defined by m_0, z_0, and $x_{j_i} \bmod q_{m_i - 1}$ for $0 \leq i \leq s$. There are at most $2n \prod_{i=0}^{p} q_i$ such factors.

Second case: $m_0 - 1 > p$. Then either layer v_{z_0} is non-zero, and it is determined by j_0 and some $0 \leq y < n$ such that $x_{j_0} + y = 0 \pmod{q_{m_0 - 1}}$, or layer v_{z_0} is zero, and we set $y = n$. The factor v is entirely defined by j_0, z_0, y, and $x_{j_i} \bmod q_{m_i - 1}$ for $1 \leq i \leq s$. As $m_1 \leq p$, there are at most $(d-1)n(n+1) \prod_{i=0}^{p-1} q_i$ such factors.

Summing both numbers, we find that

$$P(n) \leq n(2q_p + (d-1)(n+1)) \prod_{i=0}^{p-1} q_i \leq 2dn^2 \prod_{i=0}^{p-1} q_i \ .$$

We can now choose the sequence (r_i). Let (A_i) be any nondecreasing sequence of integers tending to $+\infty$ (as slowly as we want). We define (r_i) inductively, starting with $r_0 = 1$. Assume r_i is constructed for $i < m$, so that $q_j = \prod_{i=0}^{j-1} r_i$ is defined for $j \leq m$. Let then r_m be the smallest integer such that $r_m > r_{m-1}$ and $A_{q_m r_m} \geq 2d \prod_{i=0}^{m} q_i$.

We then have $P(n) \leq n^2 A_{q_p} \leq n^2 A_n$ for all $n \geq q_2$. Taking $A_i = \lfloor \sqrt{i} \rfloor$ for instance, we get $P(n) = o(n^3)$.

References

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