Automorphism groups of subshifts via group extensions

(joint work with Ville Salo)

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Basic notions from group theory

Given groups G, H, K we say G is a **group extension of H by K** iff

$$\exists \text{ short exact sequence} \quad \mathbf{1} \to K \xrightarrow{\imath} G \xrightarrow{\phi} H \to \mathbf{1}$$

i.e. $\imath : K \hookrightarrow G$ (inclusion) and $\phi : G \twoheadrightarrow H$ (surjection) are homomorphisms, $\imath(K) = \ker(\phi)$.

If there exists a **section** $\tau : H \to G$ (homomorphism) such that $\phi \circ \tau = \mathrm{Id}_H$, we say the extension is **(right) split**.

Lemma: Let G, H, K be groups.

(1) If $\mathbf{1} \to K \xrightarrow{\imath} G \xrightarrow{\phi} H \to \mathbf{1}$ is **right split** then G is (isomorphic to) a **semi-direct product** $K \rtimes_\psi H$.

(2) If furthermore $\tau(H) \trianglelefteq G$ then $G \cong K \times H$ is a **direct product**.

Basic ways of forming new groups from G and H:

$G \times H = \{(g, h) \mid g \in G,\ h \in H\}$ with product $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

$G \rtimes_\psi H = \{(g, h) \mid g \in G,\ h \in H\}$ with product $(g_1, h_1) \cdot (g_2, h_2) = (g_1\, \psi_{h_1}(g_2), h_1 h_2)$,
where H acts on G (from the left) by group automorphisms $\psi : H \times G \to G$ such that $g \mapsto \psi_h(g) = \psi(h, g)$.

Basic notions from symbolic dynamics

$\mathcal{A}$ some finite (discrete) **alphabet**

G a countable, discrete and **finitely generated (f.g.) group**

$\sigma : \; G \times \mathcal{A}^G \to \mathcal{A}^G$ (left) **shift action** of G on the full shift $\mathcal{A}^G$ (homeomorphisms)

$(g, x) \mapsto \sigma_g(x)$ where $\forall h \in G$: $\big(\sigma_g(x)\big)_h := x_{g^{-1}h}$

G **subshift**: $X \subseteq \mathcal{A}^G$ shift invariant, closed subset

given by a family of forbidden patterns $\mathcal{F} \subseteq \bigcup_{F \subsetneq G \text{ finite}} \mathcal{A}^F$ on finite shapes such that

$$\mathsf{X}_{\mathcal{F}} := \left\{ x \in \mathcal{A}^G \;\middle|\; \forall\, F \subsetneq G \text{ finite} : \; x|_F \notin \mathcal{F} \right\}$$

X is a G-**SFT** $:\Longleftrightarrow \exists\, \mathcal{F} \subseteq \bigcup_{F \subsetneq G \text{ finite}} \mathcal{A}^F$ with $|\mathcal{F}| < \infty$ and $X = \mathsf{X}_{\mathcal{F}}$ (local rules)

X is a G-**sofic shift** $:\Longleftrightarrow X$ is a (subshift) factor of some G-SFT

The **automorphism group** of X:

$$\mathrm{Aut}(X) := \{\phi : \; X \to X \text{ homeomorphism} \mid \forall\, g \in G : \; \sigma_g \circ \phi = \phi \circ \sigma_g\}$$

Some known results about $\mathrm{Aut}(X)$ for (mixing) $\mathbb{Z}$-SFTs X

Principal tool in all proofs: Automorphisms given by **Marker constructions**

- Non-trivial, mixing $\mathbb{Z}$-SFTs have **positive entropy** (large flexibility for constructions)
- $\mathrm{Aut}(X)$ is countable, non-abelian, with **center isomorphic to** $\mathbb{Z}$ (powers of the shift map)
- $\mathrm{Aut}(X)$ is **huge**. In particular contains isomorphic copies of every finite group, $\mathbb{Z}$, the free group on countably many generators and countable direct sums of such groups
- $\mathrm{Aut}(X)$ is **not amenable**
- $\mathrm{Aut}(X)$ is **residually finite** (use denseness of periodic points and periodic point representation)
- $\mathrm{Aut}(X)$ has **decidable word problem**

(BLR 1988, KR 1990, etc.)

Qualitatively all Aut-groups of mixing $\mathbb{Z}$-SFTs look (more or less) the same

Those results extend to the setting of **mixing** $\mathbb{Z}$ **sofic shifts**

A general philosophy for studying $\mathrm{Aut}(X)$

Automorphisms do preserve both the **topological** as well as the **dynamical** structure

(e.g. orbits go to orbits, periodic points to periodic points, isolated points to isolated points etc.)

Metatheorem: X some G-subshift, $Y \subsetneq X$ a subset **defined uniquely** by "good" topological and dynamical properties, then every automorphism $\phi \in \mathrm{Aut}(X)$ preserves Y, i.e. the **restriction homomorphism** $\phi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ is well-defined with $\phi|_Y \in \mathrm{Aut}(Y)$.

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This works in particular for $Y = \mathrm{Fix}_H(X) = \{x \in X \mid \mathrm{Stab}(x) = H\}$ (set of H-periodic points)

$Y = \mathcal{T}(X)$ (set of transitive points)

$Y = \Omega(X)$ (set of non-wandering points)

$Y = X'$ (Cantor-Bendixson derivative)

$Y =$ set of points $x \in X$ whose language $\mathcal{L}(x)$ is a maximal (resp. minimal) element of the subpattern poset (ordered by inclusion) of X (e.g. transitive points or fixed points)

(obviously also works for the complement of each Y)

Disjoint unions of transitive subshifts

Proposition: (1) Let $X_1 \cong X_2 \cong \ldots \cong X_I$ be a family of **disjoint transitive** G-subshifts all **conjugate** one to another. Then

$$\mathrm{Aut}\Big(\bigsqcup_{i=1}^{I} X_i\Big) \cong \mathrm{Aut}(X_1)^I \rtimes S_I \ ,$$

where the symmetric group S_I acts on $\bigsqcup_{i=1}^{I} X_i$ by permuting its components.

(2) Let $\big(X_{i,j}\big)_{i,j}$ be a family of **disjoint transitive** G-subshifts where $1 \leqslant i \leqslant I$ and $1 \leqslant j \leqslant J_i$, and suppose that $X_{i,j} \cong X_{i',j'} \iff i = i'$. Then

$$\mathrm{Aut}\Big(\bigsqcup_{i,j} X_{i,j}\Big) \cong \prod_{i=1}^{I} \mathrm{Aut}(X_{i,1})^{J_i} \rtimes S_{J_i} \ .$$

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Corollary: This yields **complete description** of automorphism groups for **finite** $\mathbb{Z}$-subshifts:
Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a finite $\mathbb{Z}$-subshift. Then the shift σ is conjugate to a permutation $\pi \in S_{|X|}$.
Let $\pi = \prod_{i=1}^{I} \prod_{j=1}^{J_i} \pi_{i,j}$ be the **cycle decomposition** of π where each $\pi_{i,j}$ is an i-cycle.
Then

$$\mathrm{Aut}(X) \cong \prod_{i=1}^{I} (\mathbb{Z}_i)^{J_i} \rtimes S_{J_i} \ .$$

(similar result for finite G-subshifts, however cyclic groups $\mathbb{Z}_i$ would have to be replaced by more general finite groups)

Cantor-Bendixson rank of a subshift

X a G-subshift

Define the **Cantor-Bendixson derivative** of X

$$X' := X \setminus \text{ isolated points } = \{x \in X \mid \forall\, x \in \mathcal{U} \subseteq X \text{ clopen: } |\mathcal{U}| \geqslant 2\}$$

$X^0 := X$, $X^{\alpha+1} := (X^\alpha)'$, $X^\lambda := \bigcap_{\alpha<\lambda} X^\alpha$ (transfinite induction for ordinals α and limit ordinals λ)

The **Cantor-Bendixson rank** of X is the smallest ordinal α such that $X^{\alpha+1} = X^\alpha$

Example: $\mathcal{A} = \{0, 1\}$, $k \in \mathbb{N}_0$

$X_{\leqslant k} := \{x \in \mathcal{A}^G \mid \#_1(x) \leqslant k\}$ (subshift whose points contain at most k copies of symbol 1)

$G = \mathbb{Z}$: $X_{\leqslant 0} = \{0^\infty\}$, $X_{\leqslant 1} = \{0^\infty\} \cup \mathrm{Orb}\{0^\infty.1\,0^\infty\}$ (sunny side up shift), etc.

$\forall\, k \in \mathbb{N}$: $(X_{\leqslant k})' = X_{\leqslant k-1}$ and $(X_{\leqslant 0})' = \varnothing$ hence $X_{\leqslant k}$ has C.B.-rank $k+1$

Cantor-Bendixson derivatives and a first result for countable $\mathbb{Z}$-sofics

Theorem: Let $X \subseteq \mathcal{A}^G$ be any G-subshift, and let $Y = X'$ denote its **Cantor-Bendixson derivative**. Then the sequence

$$\mathbf{1} \to K \to \mathrm{Aut}(X) \to H \to \mathbf{1}$$

is **exact**, where $H = \{f|_Y \mid f \in \mathrm{Aut}(X)\} \leqslant \mathrm{Aut}(Y)$ and K is a direct union of subgroups of G^*-by-finite groups.

(Given a (f.g.) group G, let G^* denote the family of groups obtained from G by taking finite direct products.)

Remark: The image H of the restriction homomorphism can be **very small** inside $\mathrm{Aut}(Y)$. We do not know if the above group extension always splits (it does in all our explicit examples).

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Theorem: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a **countable $\mathbb{Z}$-sofic** shift. Then $\mathrm{Aut}(X)$ has **decidable torsion problem**, i.e. for each $f \in \mathrm{Aut}(X)$ it is decidable whether or not f generates a finite group. (already in Salo & Törmä, 2012)

(Isolated points in countable $\mathbb{Z}$-sofic are eventually periodic and X' can be computed from X using the graph presentation of X.)

Structure of $\mathrm{Aut}(X)$ for countable G-subshifts

Iterated application of the theorem on the Cantor-Bendixson derivative gives very strong structural results about $\mathrm{Aut}(X)$.

Definition: Let $\mathcal{G}$ be a class of (abstract) groups such that

- $\mathcal{G}$ contains all **finite groups**,
- $\mathcal{G}$ is closed under taking **subgroups**,
- $\mathcal{G}$ is closed under forming **directed unions**, and
- $\mathcal{G}$ is closed under forming **group extensions**.

Then we say $\mathcal{G}$ is an **elementary class**. The **elementary closure** of a group (or family of groups) G is the smallest class of groups which is elementary and contains G. Groups in the elementary closure of G are called G-**elementary**.

Remark: Definition contains "natural" group operations and almost coincides with definition of **elementary amenable (EA)** (abelian groups not included).

Theorem: Let $X \subseteq \mathcal{A}^G$ be a countable G-subshift. Then $\mathrm{Aut}(X)$ is in the **elementary closure of** G.

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Consequences on $\mathrm{Aut}(X)$ for countable G-subshifts

Theorem: Let $X \subseteq \mathcal{A}^G$ be a countable G-subshift. Then $\mathrm{Aut}(X)$ is in the **elementary closure of** G.

Proof ingredients: C-B-derivative Theorem implies $\ker(\phi)$ is G-elementary.

$\phi_1 : G_0 \to G_1$, $\phi_2 : G_1 \to G_2$ homomorphisms, then $\ker(\phi_2 \circ \phi_1)$ is a group extension of a subgroup of $\ker(\phi_2)$ by a subgroup of $\ker(\phi_1)$.

Transfinite induction on C-B-rank of subshift, the kernel of ϕ stays in elementary closure.

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Corollary: If G is **amenable (resp. elementary amenable, torsion, locally finite)**, and $X \subseteq \mathcal{A}^G$ is a countable G-subshift, then $\mathrm{Aut}(X)$ is **amenable (resp. elementary amenable, torsion or locally finite)**.

Corollary: In particular when X is a **countable $\mathbb{Z}$-subshift**, $\mathrm{Aut}(X)$ is **elementary amenable**. (For uncountable $\mathbb{Z}$-sofics (positive entropy) $\mathrm{Aut}(X)$ is **not amenable**)

Corollary: Let $X \subseteq \mathcal{A}^G$ be a G-subshift. Then there exists a **perfect** G-subshift $Y \subseteq X$ such that $\mathrm{Aut}(X)$ is a **group extension** of $\mathrm{Aut}(Y)$ by a G-**elementary group**.

More results about countable $\mathbb{Z}$-sofics

X **uncountable, transitive** $\mathbb{Z}$-sofic shift (dense periodic points) $\Longrightarrow \mathrm{Aut}(X)$ is **residually finite**
($\mathbb{Z}$-subshifts with dense periodic points always have residually finite automorphism groups)

Proposition: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a **countable** $\mathbb{Z}$-sofic shift.

- If C-B-rank of X is 1 (i.e. $|X| < \infty$), then $\mathrm{Aut}(X)$ is **finite**.
- If C-B-rank of X is 2 (i.e. orbit closure of finitely many eventually periodic points), then $\mathrm{Aut}(X) \leqslant \mathbb{Z}^n \rtimes G$ for some $G \leqslant S_n$.
- If C-B-rank of X is at least 3, then $\mathbb{Z}^\infty \rtimes S_\infty \leqslant \mathrm{Aut}(X)$ and in particular $\mathrm{Aut}(X)$ is **not residually finite**.

(Explain $X_{\leqslant k}$ examples, mention complexity function)

The last item is a consequence of the following more general theorem:

Theorem: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a countable $\mathbb{Z}$-subshift containing an **infinite set of isolated points** in distinct orbits with the **same eventually periodic tails**. Then $\mathbb{Z}^\infty \rtimes S_\infty \leqslant \mathrm{Aut}(X)$.

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Some explicit computations of $\mathrm{Aut}(X)$ Assume $G = \mathbb{Z}$ for simplicity

$\mathrm{Aut}(X_{\leqslant 1}) = \langle \sigma \rangle \simeq \mathbb{Z}$ (unitary point is fixed by all automorphisms, only shifts on isolated points)

$\mathrm{Aut}(X_{\leqslant 2}) \simeq \mathbb{Z} \times (\mathbb{Z}^{\infty} \rtimes S_{\infty})$ (theorem applies, explain structure, not finitely generated) . . .

In principle able to determine $\mathrm{Aut}(X_{\leqslant k})$ explicitly for all $k \in \mathbb{N}$, but expressions get more and more complicated.

Proposition: The infinite periodic subshift has $\mathrm{Aut}\left(\overline{\bigcup_{n\in\mathbb{N}} \mathrm{Orb}\{(1\,0^n)^{\infty}\}}\right) \simeq \mathbb{Z} \times \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_n$

Idea of proof: $\left(\overline{\bigcup_{n\in\mathbb{N}} \mathrm{Orb}\{(1\,0^n)^{\infty}\}}\right)' = X_{\leqslant 1}$ and in this particular case our exact sequence splits with normal image, which gives $\mathbb{Z}$ in the direct product.

So we only have to determine the kernel of the restriction map.

Every automorphism preserves period, thus maps periodic orbits into themselves and also has to fix points with very large periods.

Comparison between $\mathrm{Aut}(X_{\leqslant 2})$ and $\mathrm{Aut}\big((X_{\leqslant 1})^2\big)$

Taking Cartesian products of countable subshifts (even countable $\mathbb{Z}$-sofics) makes the Aut-group much **more complicated** and in particular **destroys many algebraic properties**.

$\mathrm{Aut}(X_{\leqslant 2}) \cong \mathbb{Z} \times (\mathbb{Z}^\infty \rtimes S_\infty)$	$\not\cong$	$\mathrm{Aut}\big((X_{\leqslant 1})^2\big) \cong (\mathbb{Z}^\infty \rtimes S_\infty) \rtimes (\mathbb{Z}^\infty \rtimes S_2)$
not residually finite		not residually finite
not finitely generated (S_∞ + lemma)		**finitely generated** (3 generators)
locally virtually abelian		not locally virtually solvable
all f.g. subgroups have **polynomial growth**		**exponential growth** (free monoid on 2 generators)

$\mathrm{Aut}\big((X_{\leqslant 1})^2\big)$ already contains isomorphic copies of **all finite** and **all finitely generated abelian** groups. (huge automorphism group despite being zero entropy, countable $\mathbb{Z}$-sofic)

Remark: There is a weak partial result about when the automorphism group of a Cartesian product of G-subshifts X_1, X_2 is a (semi-direct) product of $\mathrm{Aut}(X_1)$ and $\mathrm{Aut}(X_2)$. (uses strong independence assumptions)

What remains of Ryan's theorem?

Useful (only tool) for proving non-isomorphism of automorphism groups for mixing $\mathbb{Z}$-SFTs

Proposition: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a **countable** $\mathbb{Z}$**-sofic**, then

$$\mathcal{Z}\big(\mathrm{Aut}(X)\big) \cong \mathbb{Z}^k \times {\mathbb{Z}_2}^l \qquad \text{for some } k \in \mathbb{N} \text{ and } l \in \{0,1\}$$

However, there exists $X \subseteq \mathcal{A}^{\mathbb{F}_2}$ **countable sofic shift on the free group** on 2 generators such that:

- X is faithful (i.e. $\bigcap_{x \in X} \mathrm{Stab}(x) = \mathbf{1}$)
- C-B-rank of X is 2
- $\mathrm{Aut}(X) = \mathbf{1}$