Deviations of ergodic averages for systems coming from self-affine point sets

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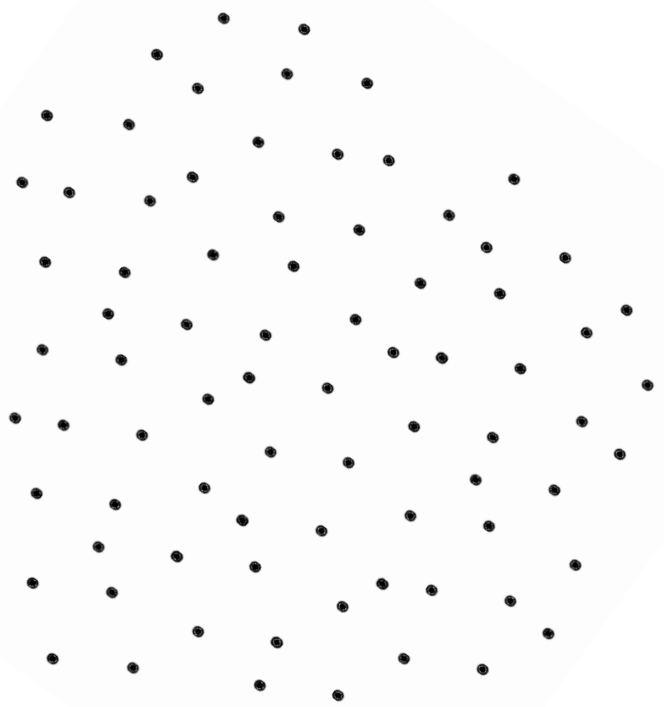
joint work with S. Schmieding



$$\Lambda \subset \mathbb{R}^d$$

Delone set

(Relatively dense, uniformly discrete)

A **cluster** or **patch** is a finite subset of Λ

Λ is **repetitive** if for every $r > 0$ there is a $R > 0$ such that any cluster in some B_r appears in every B_R

Λ has **finite local complexity** (**FLC**) if for every $R > 0$ there is a finite set L_R such that any cluster in any B_R is found in L_R. Or: **FLC** iff $\Lambda - \Lambda$ is discrete and closed.

$\varphi_t(\Lambda) = \Lambda + t \;\leftarrow$ translation of Λ by $t \in \mathbb{R}^d$

(assume $\varphi_t(\Lambda) = \Lambda$ implies $t = \bar{0}$)

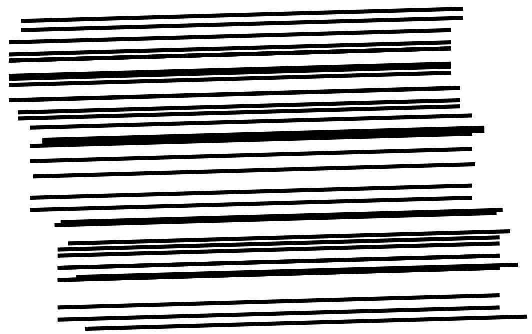
$$d(\Lambda, \Lambda') = \inf\{\varepsilon > 0 : B_{\varepsilon^{-1}} \cap \varphi_x(\Lambda) = B_{\varepsilon^{-1}} \cap \varphi_y(\Lambda')\}$$

$x, y \in B_\varepsilon$

$\Omega_\Lambda = \overline{\{\varphi_t(\Lambda),\, t \in \mathbb{R}^d\}} \leftarrow$ pattern space of Λ

Compact metric space. Foliated by orbits

Not a manifold.



Local product structure: $V \times \mathcal{C}$

$V \subset \mathbb{R}^d$ open ball

FLC

$\mathcal{C}$ totally disconnected set (usually Cantor)

$\varphi_t : \Omega_\Lambda \to \Omega_\Lambda$ Minimal, uniquely ergodic

Repetitivity Uniform cluster frequency

There exists a measure μ such that for any $f \in C(\Omega_\Lambda)$

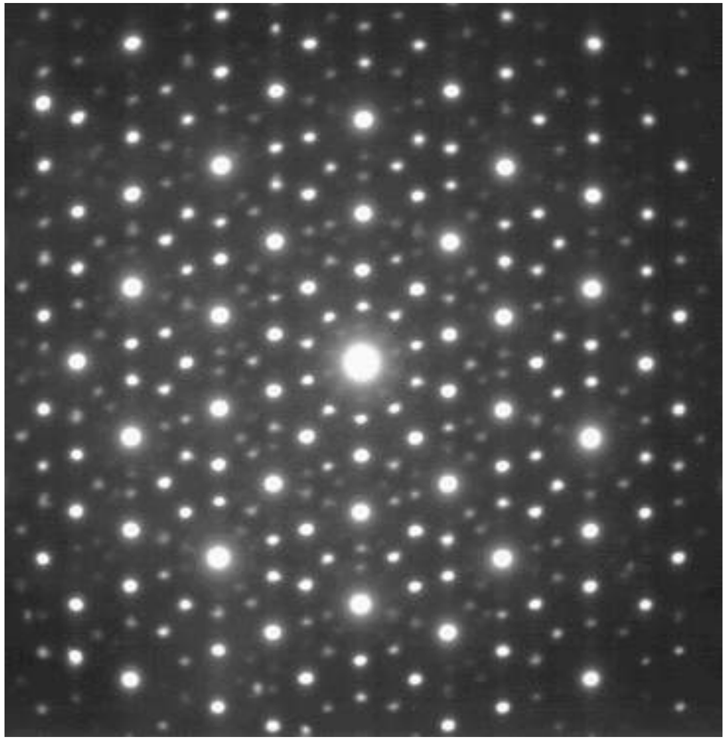
$$\frac{1}{\text{Vol}(B_T)} \int_{B_T} f \circ \varphi_t(\Lambda_0)\, dt \longrightarrow \int_{\Omega_\Lambda} f\, d\mu$$

What can we say about this convergence?

How does this grow as T gets larger?

$$\left| \int_{B_T} f \circ \varphi_t(\Lambda_0)\, dt \right|$$

Motivation: Diffraction



$\phi : \mathbb{R}^d \to \mathbb{R}$ smooth bump function

$$\rho := \phi * \sum_{x \in \Lambda} \delta_x$$

$\exists h \in C(\Omega_\Lambda)$ such that $\rho(t) = \varphi_t^* h(\Lambda)$

Dworkin's argument

$$\gamma(x) = \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(B_T)} \int_{B_t} \rho(x+t)\rho(t)\,dt$$

(Autocorrelation)

$$= \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(B_t)} \int_{B_T} \varphi_t^*[\varphi_x^* h \cdot h](\Lambda)\,dt$$

$$= \int_{\Omega_\Lambda} \varphi_x^* h \cdot h\,d\mu = (\varphi_x^* h, h)$$

$$\hat{\gamma} = \widehat{(\varphi_x^* h, h)}$$ Diffraction measure

The diffraction measure is a spectral measure and is defined by an ergodic theorem

Cohomology for pattern spaces

(following J. Kellendonk)

$f : \mathbb{R}^d \to \mathbb{R}$ is Λ-equivariant if $\exists R > 0$ s.t.

$$\varphi_x(\Lambda) \cap B_R = \varphi_y(\Lambda) \cap B_R \text{ implies } f(x) = f(y)$$

If f is Λ-equiv. there exists an $h_f \in C(\Omega_\Lambda)$ such that $f(t) = \varphi_t^* h_f(\Lambda)$

Λ equivariant forms are maps $\eta : \mathbb{R}^d \to \bigwedge \mathbb{R}^d$ which are Λ-equivariant. Denote by Δ_Λ^k the set of C^∞ Λ-equiv. k-forms.

The complex $0 \to \Delta_\Lambda^0 \xrightarrow{d} \Delta_\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \Delta_\Lambda^d$ is a subcomplex of the de Rham complex.

The Λ-equivariant cohomology spaces are

$$H^k(\Omega_\Lambda) = \frac{\ker\{d : \Delta_\Lambda^k \to \Delta_\Lambda^{k+1}\}}{\mathrm{Im}\{d : \Delta_\Lambda^{k-1} \to \Delta_\Lambda^k\}}$$

In many cases they are finite dimensional:

· Substitution tilings (Anderson-Putnam)

· Certain cut-and-project constructions (Kellendonk, Gahler, Hunton, Forrest)

Λ is of type $\mathbf{R}$ if

- There is a m.p.h. $\Phi_A : \Omega_\Lambda \to \Omega_\Lambda$ such that

$$\Phi_A \circ \varphi_t = \varphi_{At} \circ \Phi_A$$

with $A \in GL^+(d, \mathbb{R})$ an expanding matrix

Self-affine property

- $H^*(\Omega_\Lambda)$ is finite dimensional.

This holds for substitution tilings and some cut-and-project sets.

If Λ is of type **R** there is an induced map

$$\Phi_A^* : H^d(\Omega_\Lambda) \to H^d(\Omega_\Lambda)$$

with eigenvalues $\nu_1 \geq \cdots \geq \nu_r$.

Denote by $\lambda_1 \geq \cdots \geq \lambda_d > 1$ the eigenvalues of A

The *rapidly expanding subspace* $E^+ \subset H^d(\Omega_\Lambda)$ is the span of all gen. eigenvectors with eigenvalues $|\nu_i|$ satisfying

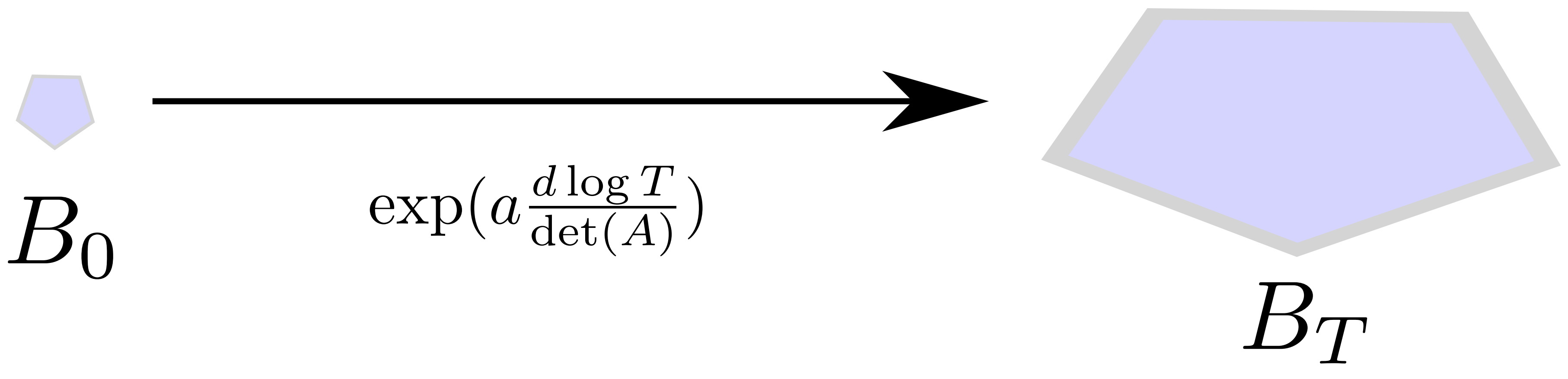
$$\frac{\log|\nu_i|}{\log|\nu_1|} \geq 1 - \frac{\log \lambda_d}{\log|\nu_1|}$$

When A is pure dilation this becomes $\frac{\log|\nu_i|}{\log|\nu_1|} \geq \frac{d-1}{d}.$

$C^{\infty}_{tlc}(\Omega_\Lambda)$ are the *transversally locally constant* functions: $h \in C^{\infty}_{tlc}(\Omega_\Lambda)$ iff $\exists\, \rho \in \Delta^0_\Lambda$ such that $\rho(t) = \varphi_t^* h(\Lambda)$

For a "nice" set B_0, let $B_T = \exp(a \frac{d \log T}{\det(A)}) B_0$ where $a \in \mathfrak{sl}(d, \mathbb{R})$ satisfies $\exp(a) = A$

As such: $\mathrm{Vol}(B_T) = \mathrm{Vol}(B_0) \cdot T^d$



Theorem (Schmieding-T): There is a ρ-dimensional space of closed, $\mathbb{R}^d$-invariant distributions $\mathcal{D}_1, \ldots, \mathcal{D}_\rho$ such that for any Lipschitz domain B_0, if $f \in C^\infty_{tlc}(\Omega_\Lambda)$ with $\mathcal{D}_i(f) = 0$ for all $i < j$ but $\mathcal{D}_j(f) \neq 0$, then if $\nu_j > \frac{\nu_1}{\lambda_d}$,

$$\left| \int_{B_T} f \circ \varphi_t(\Lambda_0)\, dt \right| \leq K T^{d \frac{\log|\nu_j|}{\log|\nu_1|}} \|f\|_\infty$$

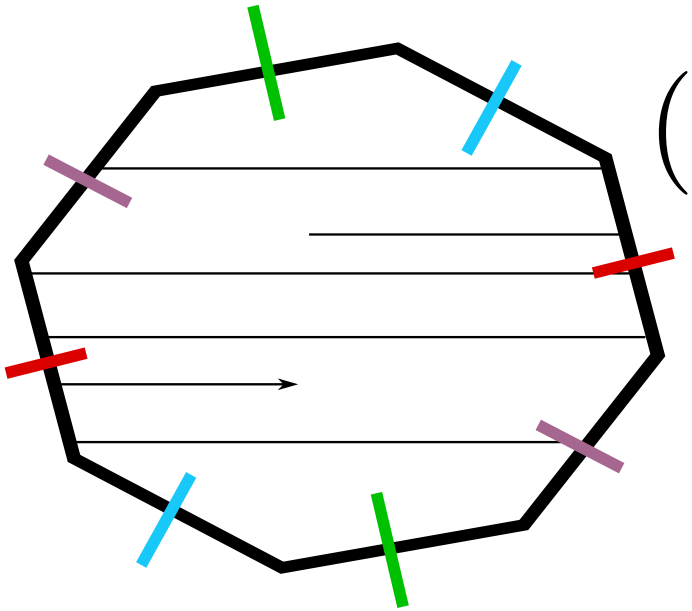
if $\nu_j = \frac{\nu_1}{\lambda_d}$,

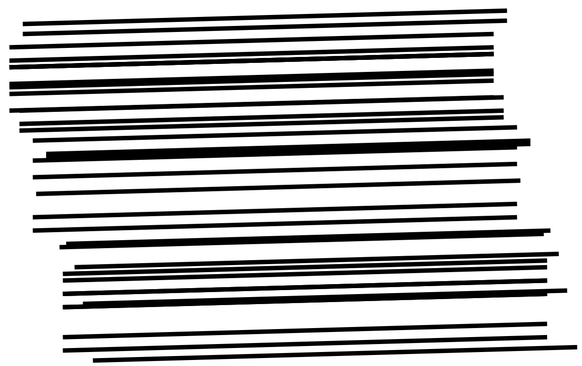
$$\left| \int_{B_T} f \circ \varphi_t(\Lambda_0)\, dt \right| \leq K (\log T) T^{d \frac{\log|\nu_j|}{\log \nu_1}} \|f\|_\infty$$

if $\mathcal{D}_i(f) = 0$ for all i, then

$$\left| \int_{B_T} f \circ \varphi_t(\Lambda_0)\, dt \right| \leq K T^{d\left(1 - \frac{\log \lambda_d}{\log \nu_1}\right)} \|f\|_\infty = \mathcal{O}(|\partial B_T|)$$

- First results using cohomology by L. Sadun
- Similar results by Bufetov-Solomyak (Self-Similar Tilings)
- Applications: diffraction, counting problems
  Solid-state models, spectral theory of Schrodinger Op's
- Inspired by the Zorich-Forni phenomena for translation flows





- $\star\mathfrak{D}_i = \mathfrak{C}_i \in (\Delta^d_\Lambda)'$ is a closed, invariant current.
- $\mathfrak{C}_1$ is the asymptotic cycle