Spectra of Pisot-cyclotomic numbers

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Real spectrum

Study of properties of the set

$$X^m(\beta) = \{\sum_{i=0}^{n} a_i\beta^i \mid a_i \in \mathcal{A}\}, \quad \mathcal{A} = \{0, 1, \ldots, m\} \subset \mathbb{N}$$

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Main questions: For which β and m is $X^m(\beta)$ uniformly discrete or relatively dense in $\mathbb{R}^+$?

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De-Jun Feng (2015): $X^{\mathcal{A}}(\beta)$ is not uniformly discrete if and only if $\beta < m + 1$ and β is not a Pisot number.

Moving to complex plane

Pisot number: An algebraic integer > 1 whose conjugates satisfy

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Note: If ω is corresponding n-th primitive root of unity, then

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From now on we will assume that $\mathcal{A} = \{\omega^n\} \cup \{0\}$

order	name	approximate value	minimal polynomial
5 or 10	τ	1.618033989	$x^2 - x - 1$
	τ^2	2.618033989	$x^2 - 3x + 1$
7 or 14	λ	2.246979604	$x^3 - 2x^2 - x + 1$
		4.048917340	$x^3 - 3x^2 - 4x - 1$
		5.048917340	$x^3 - 6x^2 + 5x - 1$
		20.44264896	$x^3 - 20x^2 - 9x - 1$
		21.44264896	$x^3 - 23x^2 + 34x - 13$
8	δ	2.414213562	$x^2 - 2x - 1$
		3.414213562	$x^2 - 4x + 2$
9 or 18	κ	2.879385242	$x^3 - 3x^2 + 1$
		7.290859369	$x^3 - 6x^2 - 9x - 3$
		8.290859369	$x^3 - 9x^2 + 6x - 1$
12	μ	2.732050808	$x^2 - 2x - 2$
		3.732050808	$x^2 - 4x + 1$

Table: Pisot cyclotomic numbers of degree 2 and 3

Interesting properties

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- Relative density: $\exists R$ s.t. each ball of certain diameter R contains a point of $X^{\mathcal{A}}(\beta)$

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- Properties of Voronoi tiling
- (C&P model)

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Let β be Pisot-cyclotomic of order n and let $\mathcal{B} \subset \mathbb{Q}(\omega)$. Then $X^{\mathcal{B}}(\beta)$ is uniformly discrete.

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Proof.

We have $\mathbb{Q}(\omega) = \{a + b\beta \mid a, b \in \mathbb{Q}(\beta)\}$

$$X^{\mathcal{B}}(\beta) \subset X^{\mathcal{B}_1}(\beta) + \omega X^{\mathcal{B}_2}(\beta) \quad \text{with} \quad \mathcal{B}_{1,2} \subset \mathbb{Q}(\beta).$$

Here $X^{\mathcal{B}_{1,2}}(\beta)$ are uniformly discrete sets. $\square$

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$$K(\beta, \mathcal{A}) := \{\sum_{i=0}^{+\infty} a_i \beta^{-i} \mid a_i \in \mathcal{A}\}.$$

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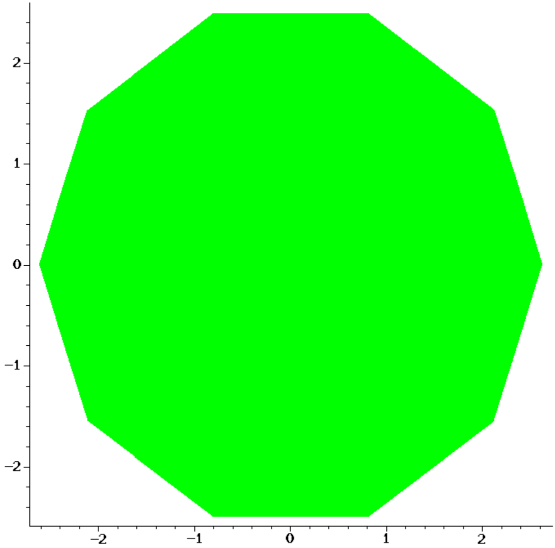
Note that $K(\beta, \mathcal{A})$ is the unique compact set satisfying

$$K(\beta, \mathcal{A}) = \bigcup_{a \in \mathcal{A}} \beta^{-1} K + a$$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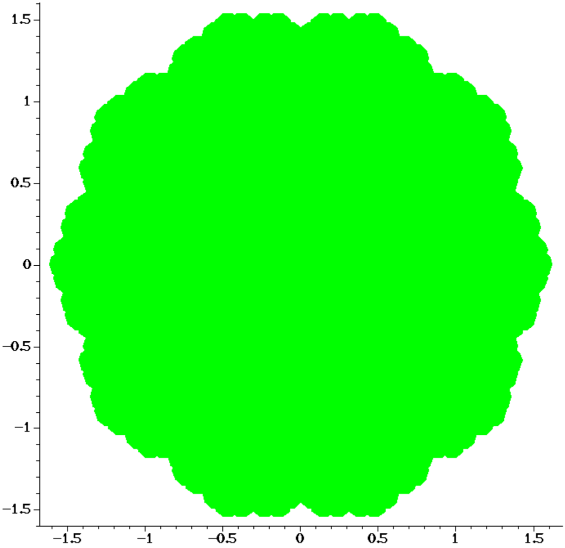
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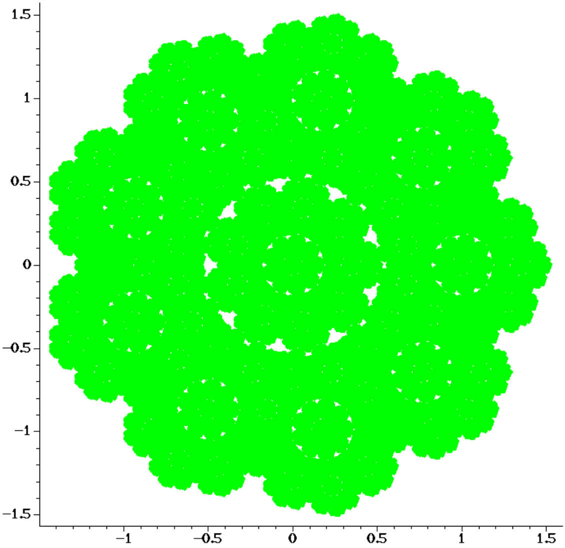
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8	δ	$x^2 - 2x - 1$	Relatively dense
9	κ	$x^3 - 3x^2 + 1$	Not relatively dense
18	κ	$x^3 - 3x^2 + 1$	Relatively dense
12	μ	$x^2 - 2x - 2$	Relatively dense

Table: Pisot cyclotomic numbers & relative density

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Theorem

Let $X^{\mathcal{A}}(\beta)$ (β and $\mathcal{A}$ arbitrary) be a discrete set. Then the following statements are equivalent:

1. $X^{\mathcal{A}}(\beta)$ *is relatively dense*
2. $0 \in \mathrm{int}(K(\beta, \mathcal{A}))$
3. *Every $z \in \mathbb{C}$ has a representation of the form* $z = \sum_{i=-\infty}^{N} a_i \beta^i$ *with* $a_i \in \mathcal{A}$

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Result of Y. Herreros, 1991: Classification of our $(\beta, \mathcal{A})$ according to property 3

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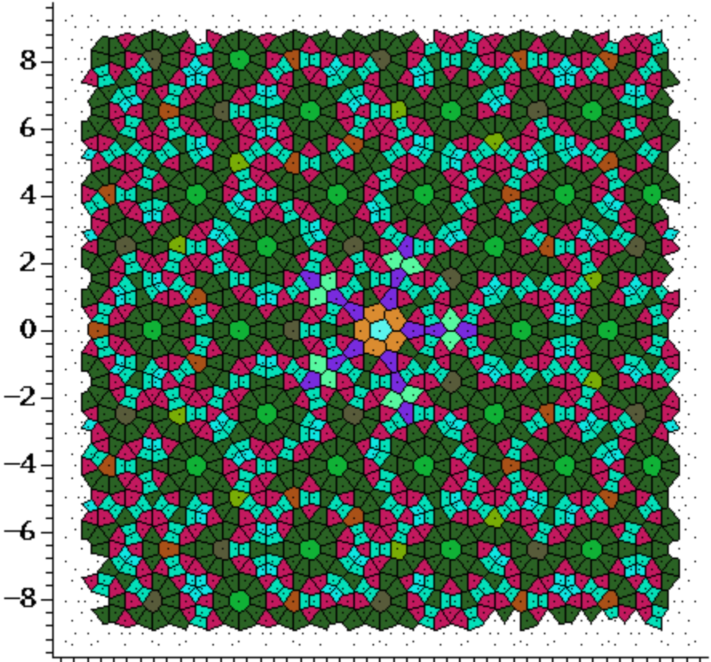
We can study

- number of tiles (up to symmetries)
- their radius (distance from the center to the farthest point)
- density of a particular tile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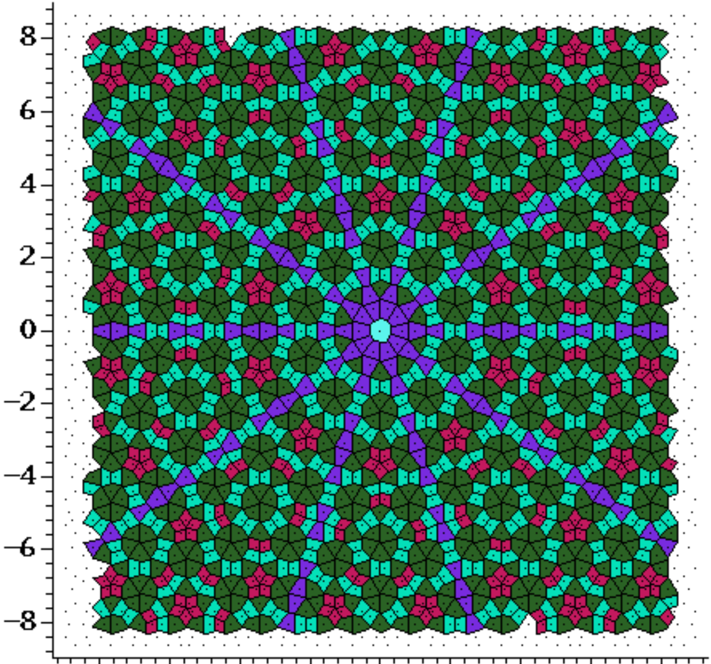
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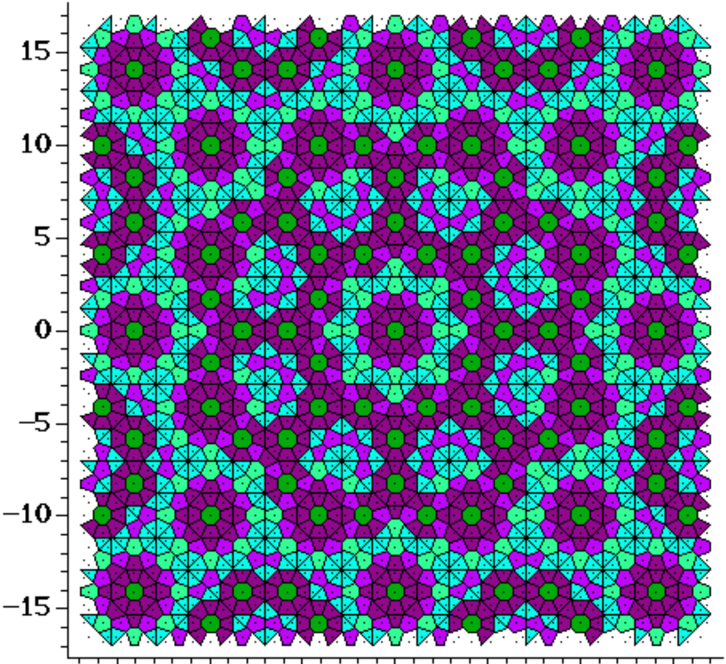

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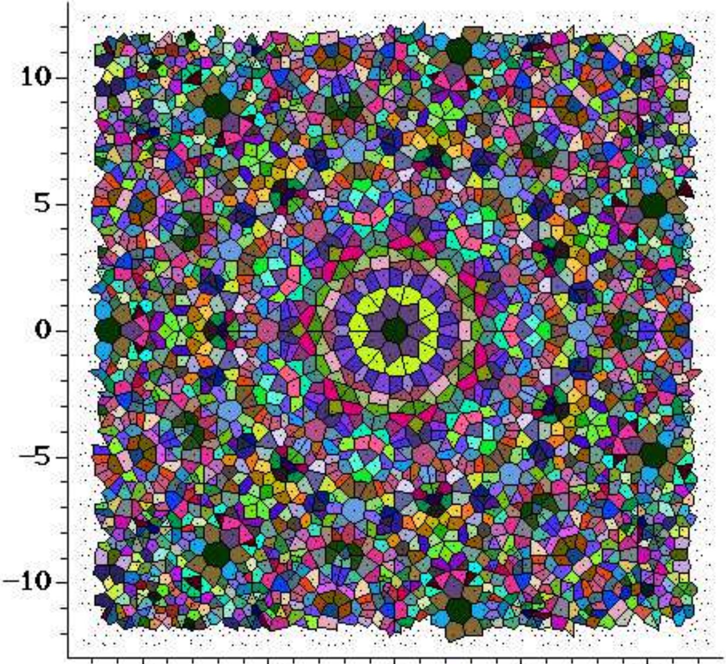
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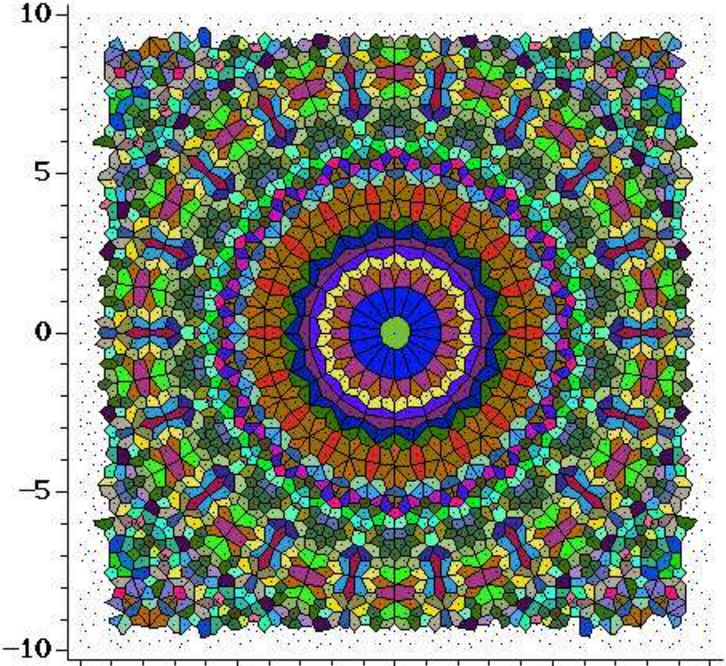
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It is possible to compute the number of tiles

Symmetric	β	Lower Bound	Upper Bound
5	τ	12	12
10	τ	5	5
10	τ^2	5	11
7	λ	201	2^{4438}
14	λ	189	2^{6594}
8	δ	5	7
18	κ	154	2^{132}
12	μ	104	2^{792}

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Voilà

Thank you for your attention